MAC 2313 Nov 19, 2020
Exam III Prof. S. Hudson

Each problem is 10 points except the TF.

1) Evaluate this integral by reversing the order of integration: $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$. Include a picture of the region R.

2) Evaluate $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} y\, dy\, dx$ by converting to polar coordinates.

3) Find the centroid of the region between the x-axis and the arch $y = \sin x$ $0 \le x \le \pi$.

4) a) Solve the system $u = 3x + 2y$, $v = x - y$ for x and y in terms of u and v. Then find the Jacobian $J = \partial(x, y)/\partial(u, v)$.

b) Find the image under the transformation in part a) of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

5) Evaluate the line integral $\int_C 5dx + (y + z)dy - xdz$ along the curve $x = 2t$, $y = 3t$, $z = t^2$ from $(0, 0, 0)$ to $(2, 3, 1)$.

6) Let $\mathbf{F} = \langle y^2e^z, 2xye^z, x^2e^z\rangle$. 6a) Is $\mathbf{F}$ conservative ? Show work or reasoning.

b) Compute div $\mathbf{F}$ (the divergence, sometimes called the flux density).

7) Compute $\int\int_R x + y^2\, dA$, where R is above the x-axis and inside $x^2 + y^2 = 1$.

8) [20pts] True-False. Assume all functions mentioned below are differentiable.

If S is a sphere with density $\delta(x, y, z)$, then center of mass of S must lie in S.

If a level curve of $f(x, y)$ and a level curve of $h(x, y)$ both pass through the point $(3, 1)$ then $\nabla f(3, 1)$ is a scalar multiple of $\nabla h(3, 1)$.

A standard formula for work along a curve is $W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

A standard formula for the flux across a curve is $\Phi = \int_C \mathbf{F} \cdot \mathbf{n}\, ds$.

The average value of $f(x, y, z) = x^2y^2z$ over the spherical region $\rho \le 5$ is zero.

If the density of a solid G satisfies $1/2 \le \delta(x, y, z) \le 2$ then the mass of G is less than its volume.

One formula relating the coordinate systems is $z/\rho = \cos\theta$.

$\int_0^\pi \int_0^{\pi/2} \int_0^1 \rho\, d\rho\, d\theta\, d\phi = 2 \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho\, d\rho\, d\theta\, d\phi$.

$\int_0^2 \int_0^4 5 + \frac{e^x}{e^y+1}\, dx\, dy = 8/3$.

Lagrange multipliers are mainly used to solve optimization problems.

9) Prove Thm 13.7.2 as done in the text: if f is differentiable at (x_0, y_0) then the tangent plane to the graph $z = f$ at (x_0, y_0) is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Remarks, Answers: The average was 53 / 100 which is rather low, with high scores of 77 and 69. The average result on the proof was only approx 20% with many left blank. Please prepare more for the proofs on the final exam! Here is a rough scale for Exam 3:

A's 63 to 100
B's 53 to 62
C's 43 to 52
D's 33 to 42

The average semester average, not yet including HW, is approx 60% with the following rough scale. I expect the scale will be similar in December, if the HW and Final exam scores hold no surprises.

A's 68 to 100
B's 58 to 67
C's 48 to 57
D's 38 to 47

1) $\int_0^\pi \frac{\sin y}{y} \int_0^y \, dx \, dy = \cdots = \int_0^\pi \sin y \, dy = 2$. You might get the last step from memory rather than the FTC.

2) 16/3, from $\int_0^\pi \int_0^2 r \sin\theta \; r \; dr \; d\theta$. One common mistake was $\int_0^{2\pi}$ but since $y \geq 0$ you should not include quadrants 3 or 4. A quick sketch may help you avoid such mistakes. Also, many people omitted $y = r\sin\theta$ from the integrand, and calculated the area of the semi-circle by mistake.

If you got a negative answer to problem 1 or 2, and kept it, you should consider giving each answer a "sanity check", to catch obvious mistakes.

3) Answer = $(\pi/2, \pi/8)$. Set $\delta = 1$ for simplicity (so, just omit it). Then the mass M is the area $A = \int_0^\pi \int_0^{\sin x} dy \; dx = \cdots = 2$. You can use a single integral for this if you like. Similarly, get $M_x = \int_0^\pi \int_0^{\sin x} y \; dy \; dx = \pi/4$ so that $\overline{y} = M_x/M = \pi/8$. You should draw a picture to see quickly that $\overline{x} = \pi/2$ (by symmetry), but it is also OK to compute from M_y/M.

4a) Use $u + 2v = 5x$ to get $x = (u + 2v)/5$, and then get $y = (u - 3v)/5$. Other methods are OK. Then $J = -1/5$.

4b) Method 1: Map (transform) the 3 line segments in the original triangle separately. The x-axis, $y = 0$ becomes $u - 3v = 0$, or $v = u/3$. Graph this line just like you would graph $y = x/3$. Likewise $x = 0$ becomes $v = -u/2$ and $x + y = 1$ becomes $2u - v = 5$. These lines bound a triangle in the uv-plane with corners at (0,0) and (3,1) and (2,-1). The first triangle is mapped to the second one (you can check this fact if you like, but that is not required).

Method 2: Based on similar examples you might guess that the answer will be a triangle. You can find the new corners easily by plugging in the old ones. For example, plug in $(x, y) = (1, 0)$ to get $(u, v) = (3, 1)$, etc. This gives you the new triangle a bit faster than Method 1. But a warning - Method 2 will fail in nonlinear examples.

5) Replace dx with $\frac{dx}{dt}\ dt$, etc. This is a common method throughout Ch.16, used with ds and $d\mathbf{r}$, etc, similar to u-substitution. Get $\int_0^1 (5)(2) + (3t + t^2)(3) - (2t)(2t)\ dt = 10 + 9/2 - 1/3$. Further simplification is not required. The limits of integration can be deduced easily from the endpoints. For example, if $2t = x = 2$ then $t = 1$.

A common mistake was to replace the given curve by a line segment, but unless the vector field is conservative that is a mistake. It is OK to use notation like $\int_C \mathbf{F} \cdot d\mathbf{r}$, if you prefer that, but it is not needed.

6) Not conservative because $N_z \neq P_y$. The divergence is $M_x + N_y + P_z = 0 + 2xe^z + x^2e^z$.

7) Sketch the region, a semi-circle. Then $\int\int_R x\ dA = 0$ by symmetry. So, delete the x, to save time (optional but suggested). The graph suggests polar coordinates, $\int\int_R y^2\ dA = \int_0^1 r^3\ dr\ \int_0^\pi \sin^2\theta\ d\theta = (1/4)(\pi/2) = \pi/8$.

Using rectangular coordinates here is a strategic mistake. The limits and the calculations are harder, probably requiring a trig substitution, at least. Nobody got it quite right that way.

8) TFTTT FTTFT.

Two of these are about symmetry. Observe that $\int_{-a}^{a} z\ dz = 0$ and $\int_0^{\pi/2}\ d\phi = (1/2)\int_0^\pi\ d\phi$. Part 9 fails a "sanity check" since the integral is at least $(2)(4)(5) = 40$.

9) See the text. It seems many people didn't study this. Set $F(x, y, z) = f(x, y) - z$. The graph of f is a level surface of F (where $F = 0$). The previous theorem in the book shows that the tangent line is $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$, which gives the equation we are trying to prove (because $F_z = -1$ and $z_0 = f(x_0, y_0)$, etc).