MAC 2313 Dec 11, 2017
Final Exam Key Prof. S. Hudson

These are 10 points each unless noted.

1) Let $\mathbf{F} = \langle x^2 - 3z, e^y, y^2 - z \rangle$. Compute div $\mathbf{F}$ and curl $\mathbf{F}$.

2) Express this integral as an equivalent integral with the order of integration reversed, $\int_0^4 \int_{\sqrt{x}}^2 f(x, y)\, dy\, dx$. You do not have to evaluate the integral.

3) Find a unit vector in the direction in which $f$ increases most rapidly at $P$, and find the rate of change of $f$ at $P$ in that direction; $f(x, y) = \cos(x - 2y)$; $P(2\pi/3, \pi/4)$.

4) Find the absolute maximum and minimum values of $f(x, y) = 4x^3 + y^2$ on the domain $2x^2 + y^2 = 1$, and where they occur. Label your final answers clearly. Small hints: 1) the min value is negative and 2) this is meant to be mainly a Lagrange problem.

5) Let $\sigma$ be the surface of the solid box bounded by the coordinate planes and $x = 3$, $y = 2$ and $z = 3$, with outward orientation. Use the Divergence theorem to compute the flux across $\sigma$ of

$$\mathbf{F} = \langle x^2 + y^2, e^x, y^3 - 2z \rangle$$

6) Use a Jacobian and a change variables to evaluate $\int\int_R \frac{x-y}{x+y}\, dA$, where $R$ is bounded by $x - y = 0$, $x - y = 2$, $x + y = 1$ and $x + y = 5$.

7) Let $\sigma$ be the part of the surface $x^2+y^2+z^2 = 4$ above $z = 1$. Compute $\int\int_\sigma (x^2+y^2)z\, dS$ and simplify.

8) [20pts] True-False:

Stoke's Thm is a standard method for computing the work done on a particle moving on a simple closed curve.

Lagrange multipliers are used to solve optimization problems.

The flux of an inverse-square field across the surface of any cube in $R^3$ is zero.

$\mathbf{F}(x, y) = \langle y^2, 2xy \rangle$ is conservative on $R^2$.

The Laplace operator is often written as $\nabla \cdot \nabla$.

Curl $\mathbf{F}$ is often written as $\nabla \cdot \mathbf{F}$.

Div (curl $\mathbf{F}$) = 0.

The set in $R^2$ where $|x| + |y| > 1$ is simply-connected.

If $\mathbf{F}$ is a vector field and $\nabla\phi = \mathbf{F}$, then $\phi$ is a potential function for $\mathbf{F}$.

If $\mathbf{r}(s)$ is parametrized by arc length, then $\kappa = ||\mathbf{r}'||$.

9) Choose ONE proof.

a) State and prove the formula in Thm.11.3.3 that relates the dot product of two vectors to the angle $\theta$ between them.

b) State and prove Thm.13.6.6, about the direction of a normal vector to a level curve of $f(x, y)$. You may include a picture, but also explain the ideas with words and short calculations.

**Remarks and Answers:** The average among the top 23 was approx 71, with high scores of 98 and 89. This is a good result. The grades were similar on all questions, but lower on 4) and 7) (approx 48%). I do not set a separate scale for the final, but will scale the combined grades asap.

1) Div $\mathbf{F} = 2x + e^y - 1$. Curl $\mathbf{F} = \langle 2y, -3, 0\rangle$.

2) $\int_0^2 \int_0^{y^2} f(x, y)\ dy\ dx$. As usual, a picture is strongly advised.

3) The direction is $\mathbf{u} = 5^{-1/2}\langle -1, 2\rangle$ (normalize $\nabla f$). The rate is $||\nabla f|| = \sqrt{5}/2$.

For full credit, you need to identify this answer as 'the rate' or perhaps as $D_{\mathbf{u}}f(P)$, etc, not just as $||\nabla f||$, which is part of the previous calculation. It is a good habit to circle and label your final answers, though I do not usually require that.

4) The max is $\sqrt{2}$ at $(\sqrt{1/2}, 0)$. The min is $-\sqrt{2}$ at $(-\sqrt{1/2}, 0)$. It is fairly easy to miss the possibility that $y = 0$, and deduce incorrectly that the min is positive, hence the hint.

I gave partial credit for finding the other critical points, which you should find anyway, of course. Namely, $(1/3, \pm\sqrt{7/9})$ and $(0, \pm 1)$, but these are not the extrema.

I actually meant for the domain to be $2x^2 + y^2 \leq 1$ (two-dimensional) but then the answer would be the same. The interior critical point $(0, 0)$ is not an extrema.

5) Integrate div $\mathbf{F} = 2x - 2$ to get $\Phi = 18$.

6) $\int_1^5 \int_0^2 (u/v)(1/2)\ du\ dv = \ln(5)$.

7) $9\pi$. Start using $dxdy$ with the term $z\sqrt{1 + f_x^2 + f_y^2}$ eventually simplifying to 2. Then switch to $rdrd\theta$, with a fairly easy integral. It should also be possible to start by parametrizing the surface but it seems harder that way.

8) TTFTT FTFTF

9) Most people chose to prove $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos\theta$ and did OK. Many people lost a few points for lack of explanation. I'd expect most of these in a good proof:

* State the theorem first and explain what $\theta$ is.
* Draw a picture, the triangle spanned by $\mathbf{u}$ and $\mathbf{v}$.
* Mention the Law of Cosines near the start.
* Explain briefly the calculations, perhaps noting that $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$ and/or that $\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}$.