MAC 2313 Dec 6, 2018
Final Exam Prof. S. Hudson

1) [15 pts total] Short answers. 1a) Find the volume of the tetrahedron with vertices $P(-1,2,0)$, $Q(2,1,-3)$, $R(1,0,1)$, $S(3,-2,3)$. Hints: the volume is 1/6 of the volume of a related parallelepiped. $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -7,-9,-4 \rangle$ (typo in Hint corrected 12/7/18).

1b) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)dy\ dx$ by converting to polar coordinates.

1c) Determine whether the limit exists. If so, find its value.

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{3x^2+4y^2}$$

2) [10pts] Find the curvature $\kappa(t)$, given $\mathbf{r}(t) = e^{3t}\ \mathbf{i} + e^{-t}\ \mathbf{j}$.

3) [5 pts] Evaluate the line integral $\int_C ydx + zdy - xdz$ along the curve $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$.

4) [10 pts] Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 4x^3 + y^2$ subject to the constraint $2x^2 + y^2 = 1$.

5) [10 pts] Let $\mathbf{F}(x,y,z) = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$. Let C be the boundary of σ, going CCW when viewed from above, where σ is the portion of the plane $x+y+z = 1$ in the first octant with an upward orientation. Use Stoke's theorem to evaluate $W = \int_C \mathbf{F}\cdot\mathbf{T}ds$.

6) [10 pts] Determine whether $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$ is a conservative vector field. If so, find a potential function for it.

7) [10 pts] Compute the surface integral $\int\int_\sigma x+y+z\ dS$ where σ is the surface of the unit cube, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. Hint: integrate over each of the 6 faces separately. If you use any shortcuts, or symmetries, explain clearly.

8) [20 pts] Answer True or False. Assume $\mathbf{v}$ (etc) are arbitrary vectors in R^3.

If two lines in R^3 do not intersect, then they are parallel.

$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

If two planes with normal vectors $\mathbf{N_1}$ and $\mathbf{N_2}$ do not intersect, then $\mathbf{N_1}\times\mathbf{N_2} = \mathbf{0}$.

By definition, the binormal vector is $\mathbf{B}(t) = \mathbf{N}(t)\times\mathbf{T}(t)$.

If $\mathbf{F}$ is a conservative vector field in R^3 then curl $\mathbf{F} = \mathbf{0}$.

$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ is a smooth function of t.

If D is an open set in R^2 then every point in D is an interior point.

If f_x and f_y exist and are continuous, then $f(x,y)$ is differentiable.

For a moving particle, the unit tangent vector and the acceleration vector are parallel.

If $\mathbf{p} = \text{proj}_{\mathbf{u}}\mathbf{v}$ then $||\mathbf{p}|| \le ||\mathbf{u}||$.

9) [10 pts] Choose ONE and prove it.

a) If $\mathbf{r}(t)$ is a vector-valued function, with $||\mathbf{r}(t)|| = 1$ for all t, then $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$.

b) State and prove (explain carefully) Thm 13.6.6, which relates level curves of $f(x, y)$ to gradients. Include $\mathbf{T}$ in your proof.

c) State and prove the formulas for a_N and a_T in Thm.12.6.3, which are given in terms of $\mathbf{v}$ and $\mathbf{a}$. If you include a picture, explain it in words.

Bonus: Draw and/or discuss Fubini's counterexample to reversing the order of integration in double integrals.

Remarks and Answers: The average was approx 68 with high scores of 89 and 87. The average results were similar on all problems except for number 7 (approx 40%). I do not set a separate scale for the final exam.

1a) $V = |\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})| = 2/3$. Because of a typo in the Hint, many people got 22/3 instead, so I also gave full credit for that answer (and a few others).

1b) $\int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \pi/8$.

1c) The limit along $y = x$ is $2/7$, but along $y = 0$ is 0. No limit.

2) $\frac{12e^{2t}}{(9e^{6t}+e^{-2t})^{3/2}}$. I'd use the cross-product method for this, converting from R^2 to R^3 as usual by inserting a zero. But there are other options.

3) $\frac{1}{3} + \frac{2}{5} - \frac{3}{4} = \frac{-1}{60}$.

4) Max of $\sqrt{2}$ at $(\sqrt{1/2}, 0)$. Min of $-\sqrt{2}$ at $(-\sqrt{1/2}, 0)$. For full credit, you should find all 6 critical points in your work, including two where $x = 0$ and two where $x = 1/3$.

5) $W = \int\int_\sigma \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS = \int\int_T \langle 1, 1, 1\rangle \cdot \langle 1, 1, 1\rangle \, dxdy = 3A = 3/2$. Here, T is a triangle in the xy-plane with area 1/2. I am using the standard shortcut, that avoids normalizing $\mathbf{n} = \langle 1, 1, 1\rangle$. A few people tried to use polar coordinates, but that is a bad idea for the shapes in this example.

6) It is conservative ($f_y = g_x = -e^x \sin y$). The potential is $\phi(x, y) = e^x \cos y + K$, but you can omit the K, since we are asked for only one potential.

7) 9. Many ways to 9; here is one. Let σ_0 be the face where $x = 0$. We can let $dS = dydz$ there. We easily get $\int\int_{\sigma_0} x \, dS = 0$ and $\int\int_{\sigma_0} y \, dS = 1/2$. Since z is the same,

$\int\int_{\sigma_0} x + y + z \; dS = 1$. The other faces are pretty similar (but not all equal). Where $x = 1$, $\int\int_{\sigma_1} x+y+z \; dS = 2$. By symmetry, the other 4 faces give the same two numbers, with total $1+1+1+2+2+2 = 9$.

Several students wanted to use the Divergence Thm, but didn't get far. To do that, we'd need a vector field with $\mathbf{F} \cdot \mathbf{n} = x+y+z$ on all 6 faces, with 6 different $\mathbf{n}$'s. Finding such a $\mathbf{F}$ is possible, but awkward. This idea does not seem best.

You do not have to use the exam Hints, but people usually do better by using them.

8) FTTFT FTTFT . The 6th one is false because the graph has a cusp at (0,0). Or, to be more precise, because $\mathbf{r}'(0) = \mathbf{0}$. This messes up the definition of $\mathbf{T}$ there, so that many Chapter 12 ideas fail there.

9) See the textbook or lecture notes. Always remember to include at least a few words in every proof, enough to explain your reasoning and justify the steps. Remarks:

9a) This was "advertised" for Exam II, but not for the Final, and the results weren't very good. It should be easy if you remember to use the product rule.

9b) Chosen only once.

9c) The most popular. Many people apparently memorized the main formulas in the proof, such as $a_N = || \mathbf{a}|| \cos\theta = \frac{\mathbf{a} \cdot \mathbf{v}}{|| \mathbf{v}||}$, which was perhaps a good start. The main problem was failure to justify the equations or to even comment on them. If you use a variable such as "θ" in a proof, you need to introduce it, and usually you need to define it. I generally accepted a picture similar to the text / lecture one, plus short phrases such as "by trig" or "by an old theorem", but I did not give full credit for incorrect or non-existent explanations.

Bonus) See the link on your HW page if interested. This topic was not required.