MAC 2313 Dec 12, 2019
Final Exam Prof. S. Hudson

1) Let $\mathbf{u} = \langle 0, 3, 4\rangle$ and $\mathbf{v} = \langle 10, 11, -2\rangle$. Compute $\text{proj}_{\mathbf{v}}\mathbf{u}$ and simplify.

2) Suppose $d\mathbf{r}/dt = -t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Compute $\mathbf{r}(t)$.

3) Find the equation of the tangent plane to the surface $2z - x^2 = 0$ at the point (2,0,2).

4) [8 pts] Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$. This curve is a "sideways 8". The two tangent lines at $(0, 0)$ are $\theta = \pm\pi/4$.

5) [7 pts] Evaluate $\int_C y\ dx$ where C is the ellipse $4x^2 + 9y^2 = 36$ oriented CCW.

6) Let $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z\rangle$. 6a) Compute curl $\mathbf{F}$ and simplify.

6b) Is $\mathbf{F}$ conservative ? Show work or reasoning.

7a) Find a potential function for the vector field $\mathbf{F} = \langle \cos(y), -x\sin(y) + 2y\rangle$.

7b) Compute $\int_{(0,0)}^{(1,\pi)} \mathbf{F} \cdot d\mathbf{r}$.

8) Let S be the surface of the solid bounded by $z = 1-x^2-y^2$ and the $xy-$plane. Compute the outwards flux of $\mathbf{F} = \langle 2x^3 + y^3, y^3 + z^3, 3y^2z\rangle$ across S.

9) [15 points] Answer T or F; you do not have to explain. Assume $f(x, y)$ is a differentiable function defined on the entire plane.

If S is a triangle with density $\delta(x, y) = x^2 + y^4$, its center of mass must lie in S.
If (a, b) is a local minimum of f, then $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).
The function $f(x, y)$ must have an absolute maximum in the region $x^2 < y$.
If div $\mathbf{F} > 0$ on a cube D in R^3, then $\Phi \geq 0$ (outwards flux across the boundary).
If $f_x(0, 0)$ and $f_y(0, 0)$ both exist, then f is continuous at $(0, 0)$.

10) Circle ONE to do. If you use the back of the page, leave a note here.

a) State and prove Thm.14.5.9 which relates D_uf to ∇f.

b) State and prove the formula in Ch 14.5 that relates ∇f to a level curve of f.

c) State and prove "Green's Lemma" (about $\int_C f\ dy = \int\int_R f_x\ dA$).

Remarks and Answers: The average was approx 54 / 100, with high scores of 73 and 71. I don't think this will change the previous scale much, but I have not included the HW scores yet. The best results were on problem 2 (88%) and the worst were on 5, 7 and 8 (30% to 35%).

1) $\frac{1}{9}\mathbf{v}$.

2) $\mathbf{r}(t) = \langle 1 - t^2/2, 2 - t^2/2, 3 - t^2/2 \rangle$.

3) Using the level surface method, $\mathbf{n} = \langle -2x, 0, 2 \rangle = \langle -4, 0, 2 \rangle$. After simplifying, get $2x - z = 2$.

4) You can use Calc II methods, but I expected Calc III ones (see 15.4 Ex2). By symmetry, you can study the part in the first quadrant and multiply by 4:

$$A = 4 \int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r \; dr \; d\theta = 8 \int_0^{\pi/4} \cos 2\theta d\theta = 4.$$

A few people replaced the $r \; dr$ by $\sqrt{4\cos 2\theta} \; dr$ but the relation $r = \sqrt{4\cos 2\theta}$ is only true on the boundary of the region. Formulas like this affect the limits of integration but not the original integrand.

5) Use Green's thm with $f(x, y) = y$ (or M=f=y and N=0), so $\int_C y \; dx = -\int\int_R \; dA = -$ area of the ellipse $-\pi ab = -6\pi$. Note that πab replaces πr^2 and you can get a by setting $x = 0$, etc.

6a) curl $\mathbf{F} = \mathbf{0}$

6a) Yes. Check the usual three equations, for example $M_y = N_x$ checks because $0 = 0$, etc. Ideally you should also mention that the domain (R^3) is simply-connected.

7a) $f = x\cos(y) + y^2 + K$. The K is optional.

7b) Using 7a and "the FTC", $f(1, \pi) - f(0, 0) = \pi^2 - 1$.

8) Use the Divergence Thm to replace the surface integral by an easier triple integral:

$$\Phi = \int\int\int \text{div}\mathbf{F} \; dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6r^2 \; dz \; r \; dr \; d\theta = \cdots = \pi.$$

9) TFFTF

10) See the text or lectures.