MAC 2313 Apr 24, 2019
Final Exam Prof. S. Hudson

NAME:

Show all your work and reasoning for maximum credit. You should not have any electronic devices within reach. No hats, hoods, notes, or personal scratch paper. You may ask about any ambiguous questions or for extra paper. Hand in any extra paper you use along with your exam. These are 10 points each unless specified.

1) The acceleration of a particle in the xy-plane at time t is given, $\mathbf{a}(t) = -2t\mathbf{j}$. The velocity after 2 seconds is $\mathbf{v}(2) = 3\mathbf{i}$ and the position after 3 seconds is $\mathbf{r}(3) = 4\mathbf{j}$. Find the position after 1 second.

2) The function $f(x, y) = 8xy - x^4 - 16y^4$ has 3 critical points. One is (-1, -1/2). Find the other two, and use the Second Partials Test to classify each of the three (as a rel.max, a rel.min or a saddle point).

3) Show that this is independent of path and compute it: $\int_{(0,4)}^{(2,0)} 3x^2e^y\ dx + x^3e^y\ dy$.

4) Find the centroid of the tetrahedron in the first octant of R^3 enclosed by the coordinate planes and the plane $x + y + z = 2$.

5) Find the equation of the plane through $(1, 2, -1)$ that is perpendicular to the line of intersection of the planes $2x + y + z = 2$ and $x + 2y + z = 3$.

6) Evaluate the surface integral $\int\int_\sigma x^2y\ dS$ where σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$ and $y = 2$ and above the xy-plane.

7) Use Green's Thm to find the work done by the force field

$$\mathbf{F} = \langle e^x - y^3, \cos y + x^3\rangle$$

on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the CCW direction.

8) [20pts] True-False:

Stoke's Theorem equates a line integral with a surface integral.

The Divergence Theorem is one of the main methods for calculating work.

The Divergence Theorem is one of the main methods for calculating flux.

The proof of Green's Thm. is based heavily on the Fundamental Thm. of Calculus.

By Green's Theorem the area of R, bounded by a CCW curve C is $A = \int_C x\ dy$.

If f is a differentiable function on the closed unit disk $x^2 + y^2 \leq 1$ in R^2, then f has a critical point somewhere on this disk.

If the graph of $z = f(x, y)$ is a plane, then both f_x and f_y are constant functions.

One of the standard formulas for curvature is $\kappa(t) = ||\mathbf{T}'(t)||/||\mathbf{r}'(t)||$.

If $D_{\mathbf{u}}f(0,0) = 0$ for all unit vectors $\mathbf{u} \in R^2$ then f is a constant function.

Unless D is a closed set, every point of D is an interior point of D.

9) Circle ONE proof and do it (you can use the back, but leave a note here):

a) In R^2, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||\ ||\mathbf{v}|| \cos(\theta)$

b) As in Thm 13.6.6, ∇f at P is normal to the level curve of f through P. You can give a careful explanation, as in the text, rather than a formal proof.

Remarks and Answers: The average among the top 18 was approx 52 / 100, with high scores of 81 and 74. The lowest scores, by far, were on problem 4, the centroid problem (average of 12%). Without that problem, the final exam average would have been pretty normal. I have not yet set a scale for the semester and do not create scales for finals. They do affect the semester scale, of course.

1) This is an initial-value problem (or two). Use $\mathbf{v}(t) = -t^2\mathbf{j} + \mathbf{C} = 3\mathbf{i} + (4 - t^2)\mathbf{j}$. Then use $\mathbf{v}$ the same way, to get $\mathbf{r}(t)$, and then $\mathbf{r}(1) = -6\mathbf{i} + \frac{14}{3}\mathbf{j}$. There were very few 100% correct answers, but half the errors were simple calculation mistakes.

2) a) Find the critical points from $\nabla f = \mathbf{0}$, meaning $8y - 4x^3 = 0 = 8x - 64y^3$. After some algebra, get $x(1 - x^8) = 0$ so that $x = 0$, 1, or -1. And $y = 0$, 1/2 or -1/2 resp. b) Using D, these are saddle point, rel.max and rel.max, resp. The most common problems were with the algebra, though some people solved for x successfully with the hint and/or guesswork.

3) Check that $f_y = 3x^2e^y = f$ and also $g_x = f$ so there is a potential, with I of P. By integrating, $\phi(x, y) = x^3e^y$. The FTLI gives $\phi|_P^Q = 8 - 0 = 8$.

It is unusual but possible to skip the $f_y = g_x$ check, checking later on that $\nabla\phi = \langle f, g\rangle$. But if so the logic should be explained. It is also possible to calculate the integral by choosing a path and a parametrization, but this did not usually turn out well in practice. I did not give much partial credit for these alternative methods.

4) Few people set this up correctly. By symmetry, we can expect $\bar{x} = \bar{y} = \bar{z}$. So, we only need to compute $\bar{z} = \int\int\int_T z\ dV \div \int\int\int_T dV$, for example. The numerator is

$$\int_0^2 z \int_0^{2-z} \int_0^{2-z-y} dx\ dy\ dz = \cdots = 2/3.$$

The denominator is vol(T) = 8/6 (from a similar calculation, or from geometry) so $\bar{z} = 1/2$ and the centroid is $(1/2, 1/2, 1/2)$.

A few people started OK but made calculation errors leading to answers like $(1, 1, 1)$ or $(-1, -1, -1)$. These can't be correct, since they are not in T (which has a simple convex shape). So, check your work, or try again, or at least make a note that your answer is unlikely but you don't have enough exam time to correct it.

5) The direction of the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 1, 1\rangle \times \langle 1, 2, 1\rangle = \langle -1, -1, 3\rangle$. For the third plane, we set $\mathbf{n} = \mathbf{v}$ and get

$$-(x-1) - (y-2) + 3(z+1) = 0.$$

6) This is a fairly standard surface integral, not a special Stoke's/Divergence type (there is no $\mathbf{F}$ here). So, we need to describe σ with 2 variables. I will use x and y (with $z = f(x, y) = \sqrt{1-x^2}$). It is possible to use θ instead of x but that's probably a bit longer.

$$\int\int_R x^2 y \sqrt{1 + \frac{x^2}{1-x^2}}\, dA = \cdots = 2\int_{-1}^{1} x^2 \sqrt{\frac{1}{1-x^2}}\, dx = \cdots = \pi.$$

Some people seemed confused over the role of θ. There are two good ways to use it here, and at least one bad way. I used a trig subn at the end (such as $x = \sin\theta$ or $x = \cos\theta$). If you parametrize the surface using θ (and y) instead of x, you may set $x = \cos\theta$, but not $x = r\cos\theta$, since the radius of the cylinder is $r = 1$, and not variable. I don't think the polar formula $x = r\cos\theta$ has any good purpose here.

7) $3\pi/2$. Use $\int\int_R g_x - f_y\, dA = \cdots = \int_0^{2\pi}\int_0^1 3r^2\, r\, dr\, d\theta = \cdots = 3\pi/2$.

8) TFTTT FTTFF.

9) See the text or lecture notes. Most people chose 9a and did OK. The most common problem (as usual) was too few words.