MAC 2313 Dec 8, 2020
Final Exam Prof. S. Hudson

1) [7 pts] Let $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, \sqrt{5}t\rangle$ for $0 \le t \le 2\pi$. Find the arc length of this curve.

2) [7 pts] Find the equation of the plane through $(1, 5, -1)$ that is perpendicular to the line of intersection of the planes $x + 2z = 2$ and $x + 3y - z = -3$.

3) Find all critical points of the function $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$. Classify the critical points as either relative maxima, relative minima or saddle point.

4) Find the work done by $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, for $0 \le t \le 1$, in the direction of increasing t.

5) Integrate $G(x, y, z) = x^2$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

6) Let $\mathbf{F}(x, y, z) = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$. Let C be the boundary of σ, going CCW when viewed from above, where σ is the portion of the plane $x + y + z = 1$ in the first octant. Use Stoke's theorem to evaluate $W = \int_C \mathbf{F} \cdot \mathbf{T} ds$.

7) [20pts] Answer True or False, you do not have to explain.

It is possible for two vectors in R^3 that $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u} \times \mathbf{v}||$.

The quadratic surface $z = x^2 - y^2$ has a saddle point.

For all angles $x > 0$, $\sin^2 x = \frac{1+\cos 2x}{2}$.

In R^3, $\text{proj}_{\mathbf{j}}(\mathbf{i} + \mathbf{k}) = \mathbf{0}$.

If f is continuous on R^2, then $\int_0^4 \int_0^3 f(x, y) dy dx = \int_0^4 \int_0^3 f(z, x) dx dz$.

If D is a bounded solid contained in the first octant of R^3, then the centroid of D lies in D.

If D is an closed set in R^2 then every boundary point of D is in D.

If R is any rectangle or disk in R^2 then $\int\int_R (4 - x^2 - y^2)\ dA \le 4\pi$.

For a particle moving in the xy-plane in R^3, the acceleration vector $\mathbf{a}(t)$ and the normal vector $\mathbf{N}(t)$ are orthogonal.

If f_x and f_y exist and are continuous, then $f(x, y)$ is differentiable.

8) [8 pts] Evaluate

$$\int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x - y}{2}\ dx\ dy$$

using this transformation (a uv substitution)

$$u = \frac{2x - y}{2}, \quad v = \frac{y}{2}$$

and integrating over a region G in the uv-plane. Helpful remarks; $x = u + v$ and $y = 2v$. You can compute J. If $x = (y/2) + 1$, then $u = 1$ (typo corrected, 12pm, 12/8/20) which gives one boundary line of G.

9) Find the average value of $f(x, y) = x\cos(xy)$ over the rectangle R where $0 \le x \le \pi$ and $0 \le y \le 1$.

10) [8 pts] Choose ONE, circle it and do it.

a) State and prove Thm 1 of Ch 12.3, a formula for the dot product. Be sure to explain θ and to justify the main steps of the proof.

b) State and prove the simple form of Green's Thm on a rectangle. You can give the textbook proof, or the one from my lecture.

Bonus) [5 pts] Find the centroid of the infinite region (the unbounded region) in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. Handle the improper integrals correctly using limits.

Remarks, Answers: The average was 52 / 100 with high scores of 81 and 76. The results were rather weak on problems 2, 5, 6 and 9 (approx 40%). This might bring the semester scale down a bit, but I expect the HW scores will bring it back to roughly the semester scale on the Exam III Key.

1) $L = \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t + 5)^{1/2}\ dt = 6\pi$.

2) $-6(x-1) + 3(y-5) + 3(z+1) = 0$. Get the normal vector by taking a cross-product, $\mathbf{n} = \langle 1, 0, 2\rangle \times \langle 1, 3, -1\rangle = \langle -6, 3, 3\rangle$.

3) Set $0 = f_x = 12x - 6x^2 + 6y$ and $0 = f_y = 6y + 6x$ (so $y = -x$). Solve to get two stationary points, $x = 0, y = 0$ and $x = 1, y = -1$. Using D we see that $(0, 0)$ is a local min and $(1, -1)$ is a saddle point (see 14.7.15).

4) $\mathbf{F} \cdot \mathbf{r}' = \langle t^3, t^2, -t^3\rangle \cdot \langle 1, 2t, 1\rangle = 2t^3$, so $W = \int_0^1 2t^3\ dt = 1/2$.

5) $4\pi/3$. The cleanest method is probably a parametrization with ϕ and θ. I used the level surface approach, with $\frac{d\sigma}{dA} = \frac{||\nabla F||}{|F_z|} = 1/z$, so $\int\int_S x^2\ d\sigma = 2\int\int_R \frac{x^2}{z}\ dA$. The 2 is to include the lower hemisphere. My parameters are x and y, so $dA = dx\ dy = r\ dr\ d\theta$. Using polar coordinates, $2\int_0^{2\pi}\int_0^1 (r\cos\theta)^2\ r(1 - r^2)^{-1/2}\ drd\theta = \cdots = 4\pi/3$. The final steps are slightly messy MAC 2312 methods.

6) $W = \int\int_\sigma \text{curl}\ \mathbf{F} \cdot \mathbf{n}\ dS = \int\int_T \langle 1, 1, 1\rangle \cdot \langle 1, 1, 1\rangle\ dxdy = 3A = 3/2$. Here, T is a triangle in the xy-plane with area 1/2. I am using a standard shortcut, that avoids normalizing $\mathbf{n} = \langle 1, 1, 1\rangle$ and omits dS/dA.

7) TTFTT FTFFT

8) $J = \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 2$. And $\frac{2x-y}{2} = u$. And the new limits of integration come from the old ones (example: $x = (y/2) + 1$ becomes $u + v = v + 1$ becomes $u = 1$, etc). So the answer is $\int_0^1 \int_0^2 u \; dv \; du = 2$.

9) Get $\int_0^\pi \int_0^1 x \cos(xy) \; dy \; dx = 2$ (using $u = xy$. Using $dx \; dy$ instead is probably harder). Then divide by the area of R to get $f_{ave} = 2/\pi$.

10) See the text or lectures.

Bo) $M = \int_{-\infty}^0 e^x \; dx = 1$, $M_y = \int_{-\infty}^0 xe^x \; dx = -1$ and $M_x = \int_{-\infty}^0 \int_0^{e^x} y \; dy \; dx = 1/4$. So dividing the last two by $M = 1$, get $(-1, 1/4)$.