MAC 2313 Apr 21, 2021
Final Exam Prof. S. Hudson

The problems are 10 points each unless labeled otherwise.

1) The acceleration of a particle in the xy-plane at time t is given, $\mathbf{a}(t) = -2t\mathbf{i}$. The initial velocity is $\mathbf{v}(0) = \mathbf{i} + \mathbf{j}$ and the initial position is $\mathbf{r}(0) = 4\mathbf{j}$. Find the position after 1 second, and simplify.

2) Let $f(x, y) = 2xy - x^4 - y^4/16$ for $x, y \geq 0$. Find a relative maximum of f. Or, if it doesn't exist, explain. For a little extra credit, do the same for a relative minimum. Label your answer(s) clearly.

Algebra remark: The solutions of $t^8 = 1$ are $t = 1$ and $t = -1$.

3) A point moves along the intersection of the plane $y = 3$ with the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x at the point $(4, 3, 2)$?

4) Let $\mathbf{F} = \nabla(x^3y^2)$. So, $f(x, y) = x^3y^2$ is a potential function. Let C be a path in R^2 from (-1,1) to (1,1). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

5) [5pts] Let $w = 2x^2 + y^2$, $x = \cos t$ and $y = 3 \sin t$. Find $\frac{dw}{dt}$ when $t = \pi$. For full credit, use the Chain Rule (rather than expressing w directly in terms of t, though you may get some partial credit for that).

6) Integrate $G(x, y, z) = 4y$ over the surface given by $z = x + y^2$ for $-1 \leq x \leq 1$, $0 \leq y \leq 1$.

7) [5pts] Let $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$. Compute curl $\mathbf{F}$ and simplify.

8) Let $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$, as in Problem 7. Let C be the boundary of S, going CCW when viewed from above, where S is the portion of the plane $x + y + z = 1$ in the first octant with an upward orientation. Use Stoke's theorem to evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

9) [20pts] True-False: assume f is differentiable and C is smooth.

- If $\mathbf{F}$ is conservative and C is a closed curve, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
- Lagrange multipliers are used mainly to find directional derivatives.
- $\nabla \times \nabla f = \mathbf{0}$.
- The Divergence Theorem is one of the main methods for calculating work.
- By Green's Theorem the area of R, bounded by a CCW curve C is $A = \int_C x\ dy$.
- If f is continuous on the xy-plane then it has a maximum and a minimum value.
- If the graph of $z = f(x, y)$ is a plane, then $f_x + f_y$ is a constant function.
- If $D_{\mathbf{u}} f(0, 0) = 0$ for all unit vectors $\mathbf{u}$, then $\nabla f(0, 0) = \mathbf{0}$.
- If $\mathbf{u} \perp \mathbf{v}$ then $||\mathbf{u} + \mathbf{v}|| \geq ||\mathbf{u}||$.
- If C in R^3 has curvature $\kappa = 2$ at every point, then C is a circle.

10) Circle ONE proof and do it (you can use the back, but leave a note here):

a) In R^2, $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos(\theta)$. Include a picture with labels and enough comments.

b) [16.8.32 done in class] Suppose f is harmonic, which means $f_{xx} + f_{yy} + f_{zz} = 0$ on a domain D. Suppose $\mathbf{n}$ is the unit outward normal vector at points of S (the boundary surface of D). Use the Divergence Theorem to show that $\int\int_S \nabla f \cdot \mathbf{n} \, d\sigma = 0$.

Remarks, Answers: The average was 53 / 100 with top scores of 83 and 75. The lowest scores were on the recent topics, problems 6-8, averaging about 30%. I have not yet set a scale for the semester.

1) Briefly, $\mathbf{v}(t) = \int \mathbf{a} \, dt = -t^2\mathbf{i} + \mathbf{C}$. Get $\mathbf{C} = \mathbf{i} + \mathbf{j}$ (from $v(0) = \mathbf{i} + \mathbf{j}$). Likewise, $\mathbf{r}(t) = \int \mathbf{v} \, dt = (t - (t^3/3))\mathbf{i} + (t+4)\mathbf{j}$. So, $\mathbf{r}(1) = (2/3)\mathbf{i} + 5\mathbf{j}$.

2) Find the stationary point(s) from $0 = f_x = f_y$. A little algebra leads to $x = \pm 1$ and $y = 2x$ (see remarks below about the case of $x = 0$). Since $x \geq 0$ the only such point is (1,2). Using D we can check that this is a relative max (the answer). Since there is no other stationary point, there is no relative min.

Remark: $\nabla f = \mathbf{0}$ at the point (0,0) too. This would be a saddle point, but it is on the boundary and would not count as a relative min in any case. So, I ignored it in my answer above.

Remark on math notation: $1 + 3x$ is not $(1+3)x$. Likewise $1 + 2/16$ is not $3/16$. Division and multiplication have precedence over addition.

3) "Rate" = "Derivative". The question asks for $\frac{\partial z}{\partial x}$, or if you set $y = 3$, the basic Calculus I derivative $\frac{dz}{dx}$, which should be easy. Plug in (4,3,2) and get -2.

4) The FTLI says the Answer $= f(1,1) - f(-1,1) = 1 - (-1) = 2$.

5) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{dy}\frac{\partial y}{dt} = 4x(-\sin t) + 2y(3\cos t) = 0$ (when $t = \pi$, $\sin t = 0 = y$).

6) This answer views the surface as the graph of a function, $z = f(x,y)$, defined on a rectangle. So, $\frac{d\sigma}{dA} = j = \sqrt{1 + f_x^2 + f_y^2} = \sqrt{2 + 4y^2}$. Then

$$\int\int_S G \, d\sigma = \int_{-1}^{1}\int_0^1 4y\sqrt{2+4y^2} \, dy \, dx = \cdots = \frac{2}{3}(6^{3/2} - 2^{3/2})$$

7) Curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle 1,1,1 \rangle$.

8) Stoke's Theorem changes the question into a surface integral, $\int\int_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, d\sigma$. I'd view this as a level surface of $x + y + z$ (but the other options as also OK) and quickly get $\mathbf{n} = \frac{1}{\sqrt{3}}\langle 1,1,1 \rangle$ and $j = \sqrt{3}$. Answer $= \int\int_T 3dA = 3$ area of T (a triangle) $= 3/2$.

9) TFTFT FTTTF

10) See the text or lectures.