INTRO TO COMBINATORICS: MONDAY 06/02/2008

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Abstract. We were talking about recurrence relations last time. At the end of section 1 which is an intro to recurrence relation.

(1) Intro to Recurrence Relations Section 7.1
  (a) Linear
  (b) Non-linear
  (c) Homogeneous
  (d) Order
  (e) Coefficients and Varieties
(2) Characteristic Equations, with and without repeated Roots
(3) General Solution to Recurrence Relations
(4) Fibonacci Sequence
(5) Sections Covered Today
  (a) 7.1, 7.2, Tower of Hanoi
(6) Extra Credit Problem Assigned.

1. Chapter 7.1: Recurrence Relation

Learned that the Fibonacci sequence gives the recurrence relation $f_n = f_{n-1} + f_{n-2}$ a *nice* linear R.R. of order 2.

Fibonacci Sequence

(1) Linear
(2) Homogeneous (if $f_i = 0$ is a solution)
(3) Order 2
(4) Constant coefficients (in this case 1's are the coefficients)

The number of terms on the right hand side determines the order of the linear recurrence relation. Usually going to be pulling stuff from the right hand side. It's homogeneous [there's not another term without the f_n's, it's still considered linear, but it's not homogeneous. If every term of the sequence were zero and it satisfies the equation, then it is a homogeneous. And has constant coefficients.] The coefficients are numbers, and in this case, they're equal to 1. If there's anything with n multiply, this is not a linear equation.

This is a relatively simple recurrence relation[fib. seq.]

$f_0 = 1, f_1 = 1$

Thursday - found two solutions

$$g_n = \frac{(1+\sqrt{5})^n}{2}$$
$$h_n = \frac{(1-\sqrt{5})^n}{2}$$

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We expect (from Linear Algebra) that any linear combination of these two would also solve the linear recurrence relation. The solutions are sequences, if you take all the sequences it forms a vector space. The vector sp. has dimension 2, each sequence is defined by 2, we have two degrees of freedom, so the set of solutions form a basis for a vector space, the claim is that it spans the vector space.

We expect from linear algebra that $f_n = c_1 g_n + c_2 h_n$, basically will give us a formula for the fib seq.

What are these constants? we can calculate these constants with the initial data.

Solution. $0 = f_0 = c_1 g_0 + c_2 h_0$; so $g_0 = 1$, and for the same reason, $h_0 = 1$ with the initial data. The initial data is $f_0 = 0$, $f_1 = 1$.

Therefore

$$0 = f_0 = c_1 g_0 + c_2 h_0 = c_1 \cdot 1 + c_2 \cdot 1 = c_1 + c_2$$
$$1 = c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \frac{1-\sqrt{5}}{2}$$

The two initial conditions are already used, so don't need to worry about them.

Note that there's a reason why this system has a solution. The Vandermonde equation is guaranteed to be non-singular.

Remark 1. *The coefficient matrix is a special type of matrix called a "Vandermonde matrix" which is always non-singular, therefore we always get a unique solution for* c_1, c_2.

Simple part first basically ignore the ones first. change to a negative c1; we get $1 = c_1 \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2} = \sqrt{5} c_1$, therefore $c_1 = \frac{1}{\sqrt{5}}$, and $c_1 = \frac{-1}{\sqrt{5}}$

so conclusion is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \tag{1.1}$$

□

Note that the latter fraction is less than 1, so one of the homework exercises is to check the latter summand is less than or equal to 1/2, that is, one can just round the first summand off to compute f_n, with the formula above.

Remark 2.
1. *The formula for* f_n *works for* $n = 0$, $n = 1$ *because of the choice of* c_1 *and* c_2. *[2. if it works for the first two, since we have a recurrence relation - so* f_3 *has to be equal to* f_n *etc.]*
2. *The Recurrence relation also implies it works for* $n = 3$, *etc.*

This always happens with a reasonably nice R.R. It needs to be linear, homogeneous, constant coefficients, works out nicer if the roots are distinct of the polynomial. But even without that, a method like this works even if the roots are not distinct.[roots of the characteristic polynomial of the RR]

2. Chapter 7.2: Other Recurrence Relations

Example 1. *What type of RR. is* $D_n = (n-1)(D_{n-1} + D_{n-2})$*?*

This is a linear recurrence relation, but it does not have constant coefficients. This is homogeneous. This says that this grows to fast for a polynomial, and more like a factorial. There are two of the. Expecting two equations.

Example 2. *$D_n = nD_{n-1} + (-1)^n$: What kind of RR is this?*

This is a linear RR, non-constant coefficients, . Can't use Linear combinations like we did in the last solution method [at least not as easily] will get back to the non-linear homogeneous, ...

Example 3. *Of a non-linear RR. $a_n = a_n a_{n-1}$,*

There's a way to make this one linear, by taking the logarithm. Then it's about the sequence

$$\ln a_n = \ln a_{n-1} + \ln a_{n-2} \tag{2.1}$$

which reduces to solving a Linear RR.

What happened in Fibonacci example should happen in any reasonable example:

Theorem 1. *Let $q \neq 0$ [the characteristic polynomial of the recurrence relation], and $h_n = q^n$. Then h_n solves the recurrence relation,*

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0 \qquad a_k \neq 0,\ n \geq k \tag{2.2}$$

if and only if q solves the characteristic equation:

$$q^k - a_1 q^{k-1} - \cdots a_k = 0. \tag{2.3}$$

This part of the theorem is obvious. The second part not so much:

Theorem 2. *If there are k distinct roots, $q_1, \ldots, q_k$, then the general solution to the RR is*

$$f_n = c_1 q^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{2.4}$$

Proof of Theorem. is omitted, uses Linear Algebra that is advanced. Shows that what was done w. the fib seq. always works. □

Note that the Fibonacci starts at $k = 2$, so this exactly what we have seen.

If you can find one solution to the non-homogeneous equation. The connection between solutions to this and a differential equation?[perhaps superficial]

For ex. the Fib RR with $f_0 = 7$, or $f_1 = 8$, (of course changing the starting data will get different c_1, c_2) on can apply the above theorem.

Fib. $f_0 = 0$, $f_1 = 0$, satisfy the Fib RR, is called the trivial solution.

Example 4. *Solve*

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n; \tag{2.5}$$

with initial conditions, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$(Doesn't really effect any of the calculations.).

Solution. The characteristic equation is $2q^3 - q^2 - 2q + 1 = 0$. This is a recurrence relations of order 3. Want to imagine they've been brought over to the other side. To get the roots of this characteristic equation, we'd like the roots to be distinct. Note this method involves factoring a polynomial of degree 3. Plugging in possible roots to find a a factor of the polynomial in the characteristic equation, $q = 1$ works. Therefore $q - 1$ will be a factor. To save time this will be $(2q-1)(q+1)(q-1) = 0$, is the factorization, which gives the roots of the above characteristic equation.

Note that this might be hard in general, but here $f(1) = 0$ so $(q - 1)$ is a factor:

This gives

$$q = \frac{1}{2}, q = -1, q = 1 \tag{2.6}$$

According to our theorem, the general solution is a linear combination of these roots

$$a_n = c_1 \left(\frac{1}{2}\right)^n + c_2(1)^n + c_3(-1)^n \tag{2.7}$$

To get c_j,

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= c_{1/2} + c_2 - c_3 \\ 2 &= c_{1/4} + c_2 + c_3 \end{aligned}$$

Therefore, by LA, [skipping LA work] get $c_1 = -8/3$, $c_2 = 5/2$, $c_3 = 1/6$.

The formula for the solution of the recursion problem is

$$a_n = -\frac{8}{3}(\frac{1}{2})^n + \frac{5}{2} + \frac{1}{6}(-1)^n \tag{2.8}$$

□

3. Recurrence Relation with Repeated Roots

Suppose the RR has order 2; we expect 2 linearly independent solutions "$g_n \neq h_n$" from the the two roots of the characteristic polynomial. But if we only get one root, $g_n = q^n$. However this doesn't help get a 2 dimensional space of solutions if we just repeat this twice. This is based on experience/guesswork, by inserting a variable in front. There is a theorem that says this works but it is a little mysterious.

If we get one solution, then we can another by

$$h_n = nq^n \tag{3.1}$$

There is a theorem says this works.

An example of how this works. To get a second solution, put a n^2 in front, etc. Again a theorem says this works [proof beyond scope of this course, it involves generalized eigenvalues/vectors]

Example 5. $h_n = 4h_{n-1} - 4h_{n-2}$

Solution. The characteristic polynomial is $(q-2)^2 = 0$, then $q = 2$ is a repeated root. Therefore, $g_n = 2^n$ and $f_n = n2^n$. will be used to get a second solution. There's a theorem that says the general solution is $h_n = c_1 g_n + c_2 f_n$. □

Example 6 (The above with different initial conditions)**.** *Suppose* $h_0 = 1$*,* $h_1 = 3$*, then [routine]*

$$\begin{aligned} 1 &= c_1 + c_2 \cdot 0 \\ 2 &= c_1 \cdot 2 + c_2 \cdot 2 \end{aligned}$$

Therefore, $c_1 = 1$, $c_2 = 1/2$ and

$$h_n = 2^n + (1/2)n \cdot 2^n \tag{3.2}$$

Theorem in the book 7.2.2. "This method always works"

Theorem 3. *If the RR is linear, homogeneous, constant coefficients, order k and the characteristic polynomial has t roots, q_1, q_2,...q_t with multiplicities "s_i", then the general solution is of the following form*

$$h_n = H_n^{(1)} + H_n^{(2)} + \cdots + H_n^{(t)}$$

where

$$H_n^{(i)} = c_i q_i^n + c_2 n q_i^n + c_3 n^2 q_i^n + \cdots + c_{s_i} n^{s_i - 1} q_i^n \tag{3.3}$$

Homework: Apply this theorem to the above RR. The following section involves some non-homogeneous recurrence relations:

4. 7.3: Tower of Hanoi puzzle

Can you move the disks from pole 1 to pole 3 so that

(1) There is never a big disk over a small disk
(2) Move disks from one pole to the adjacent

Example 7. *What is the minimum number of moves required to solve the puzzle?*

Let $h_n =$ the minimum number of moves required.
What is $h_n =$?
Fairly easy to find the recurrence relation of h_n. Getting discs to pole 3. Somewhere along the way, all the discs have to be on pole 2.
Any solution

(1) Moves the top $n-1$ to pole 2, $\Rightarrow$ requires h_{n-1} moves
(2) Then moves the bottom to pole 3, $\Rightarrow$ requires 1 moves
(3) Then moves $n-1$ to pole $\square$, $\Rightarrow$ requires h_{n-1} moves

Therefore, the solution to the problem satisfies the recurrence relation $h_n = 2h_{n-1} + 1$. Which is

(1) Non-homogeneous
(2) Linear Recurrence Relation
(3) Initial conditions $h_0 = 0$ and $h_1 = 1$

We don't know the answer but we know a RR for the data. We know the starting data, if we had 0 discs, had 0 moves. If had 1 disc requires 1 move. Instead of working it out the methodical way, lets work out a few of the h's see if can guess the answer.

$$h_2 = 3$$
$$h_3 = 7$$
$$h_4 = 15$$
$$h_5 = 31$$

They're all pretty close to a power of 2. Might guess $2^n - 1$.

Are we sure this is right? We can make a guess of the solution, but we must prove it by induction, that this formula continues to be right.

Induction is going to be everywhere. There are two induction proofs posted online. Going to have to learn if you don't know how. Going to show a more methodical way to get a solution using c_1s and c_2s the way been talking about. It does involve some guess work but not as much as this does.

5. Extra Credit Problem

k hyperplanes how many regions does it split the space? In linear programming - assume no two of them are parallel, and they don't intersect at one point.

2-dimensional hyperplanes, and splitting up $\mathbb{R}^3$ into pieces, how many pieces of $\mathbb{R}^3$ do you get. Same type of problem. Extra credit for a partial result.

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