Combinatorics, The Last Lecture
S Hudson, 12/3/13

The 12/5/13 class period will be mainly review. The 12/3/13 lecture will go a bit beyond Chs 9 and 13, and will outline one of my papers from the Journal of Combinatorics. Here are some brief notes and a few exercises on that (intended to be easy, just to help you get the vocab and main ideas). Try these by 12/5/13 if possible and ask me about any problems you cannot solve.

1) Let $f(x) = 1 - x$, $f : [0, 1] \to [0, 1]$.

a) Show that if $S = (a, b) \subseteq [0, 1]$ then $|f(S)| = |S|$ (recall that $|S|$ means the length $b - a$ now, not cardinality). Roughly, this shows that f is a *measure-preserving* map. It is similar to the idea of 1-1, but not quite the same.

b) Show that if $g : [0, 1] \to [0, \infty)$ is continuous, then $\int_0^1 g(f(x))\ dx = \int_0^1 g(x)\ dx$. Roughly, this says $||g \circ f|| = ||g||$. So, the transformation $g \to g \circ f$ doesn't affect the size of the function.

c) Draw the square $D = [0, 1] \times [0, 1]$ as a subset of the xy=plane. Let $B \subset D$ ("B" is for "bad" pairs) be the points where $|x + y - 1| > 0.5$ (this should be the union of two triangles near corners of D). Check that the graph of f does not go through B. This is similar to placing rooks on a chessboard avoiding certain "bad" regions.

d) Roughly, exercises a) and c) say that f is an SDR. Think this over until you see the analogy with Ch 9 clearly.

2) Repeat parts a) and b) for the discontinuous function

$$f(x) = \begin{cases} 0.5 - x \text{ for } 0 \leq x \leq 0.5 \\ x \text{ for } 0.5 < x \leq 1 \end{cases}$$

3) Give an example of a function $f : [0, 1] \to [0, 1]$ for which both a) and b) fail. (try simple formulas such as x^2 or $sin(x)$, though one of these ideas is wrong).

4a) Let B be the bad set in 1c). $S = (0.2, 0.4)$ be a set of "men". Let $W_S = \{y \in [0, 1] : \exists x \in S, (x, y) \notin B\}$ ("the compatible women"). Show that $W(S) = [0.1, 1]$ (if you cannot solve this by algebra, it should be pretty clear from a picture).

b) With the same set B, show that for all intervals S, $|W_S| \geq |S|$. Roughly, this means the marriage condition (MC) holds. Recall from 1d) that an SDR exists for this B.

5) Problems 1 to 4 show some similarities between continuous (Calculus-style) problems and our study of SDRs in Ch. 9. To illustrate some differences, let $B = \{(x, y) \in D : x < y\}$, which is large triangle, roughly half of D, but not including the line $y = x$.

a) Show (as in 4b) that for all intervals S, $|W_S| \geq |S|$ so MC holds.

b) But there is no SDR with this B. Spend a few minutes looking for one until you are fairly convinced. I will give some extra credit if you can prove this using Calculus; it is not super-easy.

Remarks: 1) The rest of this web page was written for a 2008 class using the previous edition of the text. But some exercises have not changed, so this may still be useful. 2) See the text answers/hints first.

Chapter 9, some answers to HW:

1) See me for a picture if necessary - the graph has 6 vertices on each side. Connect x_1 to y_1, y_2, y_3 and y_6 (from blank squares in the top row). Etc. The rooks can go into squares 11 22 33 46 54 and 65. So, the matching includes edges x_1y_1, x_2y_2 etc.

2) See me for a picture if necessary. The left side has the White squares: 11, 13, 22, 26, 33, 35, 46, 55, 62, 66. The right side has the Black squares: 12, 16, 23, 25, 32, 34, 54, 56, 61, 65. Connect adjacent squares such as 11 to 12, etc. Then the matching and covering should be easy.

3) Just reverse the process used in problem 1; let the $x_i \in X$ be the rows of the board, etc.

5) I don't see how Chapter 9 helps solve this one, so I've removed it from the suggested list.

7) $p|X| \leq |\triangle| \leq p|Y|$.

10a) It is OK to start with almost any M^1 you want, but the larger it is, the less work you must do later. In general, it's good to include as many dead-end edges as possible (they will never be changed by the Matching Algorithm). For example: x_1y_1, x_5y_2, x_6y_3, x_7y_4, x_3y_5 and x_4y_7. We can also throw in x_2y_6. This is maximal because every x_i is used. We can set $S = X$, so $|S| = |M| = 7$.

My answer doesn't use the Matching Algorithm, but I'd probably need it for 10b (answered in text). Make sure you *know how* to use it!

12) 5 from inspection (there are only 5 women). I suppose you could also use Thm 9.3.3.

13) Two. One uses $e_j = j$ for all j. One uses $e_j = j + 1$ for all j (except $e_n = 1$). There are no others (prove that by induction).

15) If A_1 and x are removed, the new problem still satisfies the MC. Each LHS has been reduced by at most 1 (by removing x).

17) I've removed this from the suggested list (it doesn't require Ch 9 skills).

20) To me, this seems obvious from the definition of *stable.* [and I don't understand the hint in the back].

27) Convert this to a bipartite graph problem, omitting edges with $a_{ij} = 0$. Then it's fairly similar to the proof that a p-regular graph has a perfect matching. If S is a cover, then $\sum_{x_i \in S} \sum_j a_{ij} + \sum_{y_j \in S} \sum_i a_{ij} = |S|$, because each $\sum_j a_{ij} = 1$. Some terms a_{ij} will be double-counted (if $x_i \in S$ and $y_j \in S$). Since X is also a cover, we get a similar formula for $|X|$, but with no double-counting. So, $|S| \geq |X| = n$. This shows $c(G) = n$ and there is a perfect matching, which means we can place the rooks as desired.