MAD 4302 | Dec 12, 2013
Final Exam | Prof. S. Hudson

You have approx 2 hours. Unlabeled problems are 10 points each.

1) Express the multinomial coefficient $\binom{10}{4\ 3\ 2\ 1}$ as a product of binomial coefficients.

2) How many ways can you place 5 non-attacking rooks on an 8x8 board such that neither the first row nor the first column are empty?

3) Prove that D_n is an even number if and only if n is odd. (if you need a formula for D_n, you can ask me, but this will cost approx 4 points).

4) [20 pts] Answer True or False to each; you do not have to explain. Some of the notation below refers to power sets, Sterling numbers, partition numbers and complete graphs (Ramsey theory).

$|P(\emptyset)| = 1$

There are at least 720 numbers between 100 and 999 with 3 different digits.

If 500 numbers are chosen from $\{222,\ 223 \ldots 999\}$, then a pair sums to 1000.

If $1 \leq k \leq p-1$, then $S(p,k) = kS(p-1,k) + S(p-1,k-1)$.

$p_5 \neq 7$

There is an antichain in $P(\{1,2,3,4,5,6\})$ with exactly 21 elements.

In every bipartite graph, $c(G) = \rho(G)$.

$K_4 \to K_4, K_2, K_2$.

A generating function can be a polynomial.

$\forall n \in N,\ 2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$.

5) Choose ONE: prove it as in the text or the lectures.

a) [Thm.13.3.1] A marriage M in a bipartite graph G is maximal if and only if there is no M alternating path.

b) [Thm.8.2.5] $S(p.k)$ counts the number of partitions of a set of p elements into k identical boxes, none empty.

6) Label the columns of the preferential ranking matrix below a, b, c and d in that order (the men). Label the rows A, B, C and D in that order (the women). Apply the deferred acceptance algorithm to find a stable marriage (rk: Women-first takes a while).

$$\begin{pmatrix} (1,3) & (2,2) & (3,1) & (4,3) \\ (1,4) & (2,3) & (3,2) & (4,4) \\ (3,1) & (1,4) & (2,3) & (4,2) \\ (2,2) & (3,1) & (1,4) & (4,1) \end{pmatrix}$$

7) I will draw a bipartite graph on the board for this problem. Or you can; label 7 dots on the left a thru g and dots on the right 1 thru 8. Then draw in these edges;

a1 a2 a4 b2 **b6** c3 **c5** c6 **d4** d7 d8 **e2** **f3** g4

Define M from the 6 boldface edges. Apply the matching algorithm. If M is not maximal, find a marriage that is. Find a minimal cover S, with $|S| = c(G)$. Explain your reasoning.

8) [5 pts] Decide whether MC_2 holds for the bad set $B = [0, 0.7] \times [0, 0.4] \subset [0, 1]^2 \subset R^2$ (I will draw this on the board upon request). Justify your answer. Note: MC_2 was discussed in class on 12/3/13; it says that $|W_S| \geq |S|$ for all simple sets $S \subseteq [0, 1]$, where $|S|$ refers to length (or Lebesgue measure).

9) [5 pts] Find a maximal antichain A in the set $S = \{4, 6, 12, 16, 18, 30, 36, 40, 120\}$ partially ordered by the relation $a|b$. Show that it is maximal by finding $|A|$ chains that partition S (recall that $\{4, 12, 36\}$ is an example of a chain, because $4|12$, etc).

10) [5 pts] $r(3, 3) = ?$ Give a specific number, such as 17, with a brief explanation. No proof required.

11) [5 pts] Find a Fibonacci number that is divisible by 15. You can give a specific number, such as 150, or an expression such as f_{15}, though I doubt either of these is correct. Explain/Justify.

Bonus) [5 pts] The Warden and the Cell: A prison holds 23 prisoners in 23 cells. The Warden decides to give them a chance to go free. Once in a while, he will choose one prisoner randomly and take them to a special cell, which contains just one light switch. The prisoner can flip the switch, or not, and then must return to his own cell. The game ends when one of the prisoners declares that all 23 prisoners have been to the special room at least once. If he is right, they all go free, but if he is wrong they are all killed. No talking, peeking, or other communication is allowed after this strange game starts. But the prisoners are allowed to form a plan beforehand. Find a winning plan.

Remarks and Answers: The average was approx 64, which is fairly normal, with highs of 88 and 82. The average scores on each problem ranged from about 50% to 70%, except for # 6 (approx 90%), #8 and # 10 (both approx 40%).

1) $\binom{10}{4}\binom{6}{3}\binom{3}{2}$ is one answer.

2) If there is a rook in corner (1,1) there are $\binom{7}{4}^2 4!$ ways to place the other 4. If not, there are $7^2\binom{6}{3}^2 3!$ ways. Add these 2 numbers.

Alt. soln: ignoring the special rule, there are $\binom{8}{5}^2 5!$ ways. Leaving row 1 and col 1 empty, there are $\binom{7}{5}^2 5!$ ways, which shouldn't have been counted; subtract this from the previous number.

3) Start from the known formula, $D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots (-1)^n \frac{1}{n!})$. After distributing, we get n integers, and the first $n-2$ are clearly even. The last two take the form $\pm n \pm 1$, which is even iff n is odd. The first cases D_1 and D_2 do not quite fit this logic; but they are easy and I leave them to you.

Alt.soln: There is also a recursion formula for D_n which is pretty easy to use here, but it does require induction, which is a bit longer.

4) TFFTF FTTTT

5) See the text.

6) There is only one stable marriage: aC, bD, cA, dB. The men-first method takes about 4 steps and the women-first method takes about 11.

7) You should use the m.a. to find a path such as $g4d7$. This increases M by removing $d4$ and adding back in $g4$ and $d7$. This is maximal because it has 7 edges, and there is a cover S (consisting of the 7 letters a to g, for example) of that size.

8) It does not. Set $S = [0, 0.7]$ and deduce from the graph that $W_S = (0.4, 1]$, which is smaller.

9) $A = \{12, 16, 18, 30, 40\}$ has 5 elements. One partition into 5 chains is $\{4, 12, 36\}$, $\{6, 18\}$, $\{16\}$, $\{30, 120\}$, $\{40\}$.

10) 6, because $K_6 \to K_3, K_3$, but $K_5 \not\to K_3, K_3$ (you can use words instead, of course).

11) f_{20}. We know $f_4 = 3$ and $f_5 = 5$ both divide this, so 15 does too. A few people patiently computed all the f_n up to $f_{20} = 6765$, and got it that way. This method seems tedious and unreliable to me, but I gave full credit.

Bonus) There are answers online, for example:

http://theweeklyriddle.blogspot.com/2009/11/prisoners-and-light-switch.html.

If you'd like to think more about this first, here are 2 hints:
a) One prisoner is designated The Counter and will announce the end of the game when he is sure it is done.
b) I forgot that this puzzle is a little easier if the prisoners are told that the switch will start in the DOWN position. Try it this way first.