Induction Proofs

There are several versions of induction, which are all actually equivalent if you dig into them a bit. The most common version assumes the claim for n (perhaps for $n \geq 0$) and proves it for $n+1$. It is also OK to assume for $n-1$ and prove for n (perhaps for $n \geq 1$). You can choose either of these, whichever seems to make the notation cleaner.

The *strong form* of induction assumes the claim for all $k \leq n$ and proves it for $n+1$. Or for $k < n$ and proves it for n. Use the strong form when the claim for n is not directly related to the claim for $n-1$ (rather, to some earlier claim such as the one for $n-2$ or $n/2$, etc).

Exercise 7.3b: Show that f_n is divisible by 3 if and only if n is divisible by 4.

Proof: Taking a hint from the answer key, and the formula from 7.3a, we get $f_n = 2f_{n-2} + f_{n-3} = 3f_{n-3} + 2f_{n-4}$ (since $f_{n-2} = f_{n-3} + f_{n-4}$). This shows f_n is divisible by 3 iff f_{n-4} is divisible by 3. We can finish by induction (strong form) on n.

Basis Step: Note that $f(0) = 0$ is div by 4, but $f(1) = 1$ and $f(2) = 1$ $f(3) = 2$ are not. So, the claim is true for all $n \leq 3$. [notice that this basis step includes 4 cases instead of just 1; that's because we plan to use the strong form, and to refer to $n-4$ later on]

Induction Step: Fix $n \geq 4$. Assume the claim is true for all $0 \leq k < n$ [this is the induction hypothesis; strong form]. We must prove the claim for n. There are two cases. 1) If n is divisible by 4, then so is $k = n-4$, and $k \geq 0$, so we can apply the IH. So, f_{n-4} is divisible by 3. From paragraph 1, this shows f_n is too. If n is not divisible by 4, then neither is $k = n-4$. By the IH, f_{n-4} is not divisible by 3. From paragraph 1, this shows f_n is not either.

Exercise 7.19e): Solve [and prove] $h_n = 2h_{n-1} + 1$ for $n \geq 1$ with $h_0 = 1$.

Solution: $h_1 = 3$, $h_2 = 7$, $h_3 = 15$, $h_4 = 31$ etc. Apparently, these are all close to a power of 2. The pattern seems to be $h_n = 2^{n+1} - 1$.

Proof by induction: basis step: We have already checked the pattern holds for $0 \leq n \leq 4$ [really, we only need to mention $n = 0$ here].

Induction Step: Assume the claim is true for $n - 1$, and we will prove it for n. So, plugging into the recurrence relation, $h_n = 2h_{n-1} + 1 = 2[2^{n-1-1} - 1] + 1 = 2^{n-1} - 1$ which proves the claim for n.