Counting Non-Decreasing Sequences
S Hudson, 5/14/08

There was some disbelief today in class, about the following examples:

Ex A: How many solutions, such as $(3, 1, 2, 1)$, are there to the equation $x_1 + x_2 + x_3 + x_4 = 7$, where each x_j is a nonnegative integer ?

Ans: $C(10, 3) = 120$. This answer was generally accepted.

Ex B: How many non-decreasing sequences of length 7, such as 1112334, can be formed from the set $\{1, 2, 3, 4\}$?

Ans: $C(10, 3) = 120$. This answer was *not* generally accepted. OK...

I claimed that the set B of such sequences has the same number of elements as the set of solutions to Ex A. So, $|B| = |A| = 120$. My reasoning was based on a 1-1 correspondence, described below, and this theorem:

Theorem: If $f : A \to B$ is a 1-1 correspondence (eg 1-1 and onto), then $|A| = |B|$.

Usually, the f is described verbally, rather than by a formula. In this case, we can take any 4-tuple in A (such as $a = (3, 1, 2, 1)$) and get a unique sequence $f(a) \in B$ by starting with x_1 1's, and then x_2 2's, and so on (in the sub-example, $f(a) = 1112334$). Hopefully, this describes f clearly enough and everyone can see that it is 1-1 and onto.

What if you still don't believe ? If my plan of proof is clear enough, you should be able to point to some step you don't accept, and then I will supply more details. For example, you can ask *Why is this f onto?*. If your question goes outside my proof (what if ... ?) then it may have no reasonable answer.

Chapter Two: You still don't believe ??

To a mathematician, there is no substitute for a proof! That is how doubts are settled. But if you've given up on this one, I'll offer you two other choices. First, check the example on page 73. Second, we can look at simpler versions of Examples A and B, small enough that we can list all the solutions.

Example A2: How many non-negative solutions are there, to $x_1 + x_2 = 3$?

Example B2: How many non-decreasing sequences are there, of length 3, from $\{1, 2\}$?

List of the $C(4, 1) = 4$ solutions to A2: (0,3) (1,2) (2,1) (3,0).

List of the corresponding 4 solutions to B2: 222, 122, 112, 111.

I hope you can see the 1-1 corr more clearly in this small example, and also see that the two problems do have the same number of solutions.