MAP 2302 July 5, 2013

Exam 1 and Key Prof. S. A. Hudson

Note: there were two versions of the exam, A and B. If my name appears to be 'S. B. Hudson' at the top of your exam (instead of 'S. A. Hudson'), then you had version B. This key was written mainly for Exam A, but B was very similar, so I have merely inserted some short notes about B.

1) [10pts for a+b] For each IVP below, decide if it has zero, one, or more than one solution. Explain clearly. The third is a bit harder, so consider it extra credit.

a) $y\frac{dy}{dx} + x = 0$, $y(1) = 0$

b) $\frac{dy}{dx} + xy = 0$, $y(0) = 1$

c) $\frac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$

On Exam B, I interchanged a) and b) above, and kept c) the same. So, part a) on Exam B was $\frac{dy}{dx} + xy = 0$, $y(0) = 1$, etc.

2) [25pts] Answer True or False. You do not have to explain, unless you think the problem is ambiguous.

The method of grouping is used mainly for exact DE's.

The formula (from class and Ch.3.2) for terminal velocity is gm/k.

The substitution $y = vx$ is used mainly with linear equations.

The BVP $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 3$, has a unique solution.

Every function of the form $f(x) = (x^3 + c)e^{-3x}$ solves the DE $\frac{dy}{dx} + 3y = 3x^2e^{-3x}$.

On Exam B, I interchanged statements 1 and 3 above.

3) [15pts] Find the general solution to this DE; $4xy\ dx + (x^2 + 1)\ dy = 0$. This should be fairly easy, and I will not give a lot of partial credit. Check your work on this one !

On Exam B, the DE is $6xy\ dx + (x^2 + 1)\ dy = 0$.

4) [15pts] Start finding the general solution to this DE; $(x-2y-3)\ dx+(2x+y-1)\ dy = 0$. This may be quite long, so you can stop if / when you get to any fairly difficult integral. Explain (mostly in words) how you would finish.

On Exam B, the DE is $(x - 2y - 1)\ dx + (2x + y - 7)\ dy = 0$. The last two problems are the same on Exams A and B.

5) [15pts] Use the integrating factor $\mu(y) = y^2$ to solve this IVP (your solution may be either implicit or explicit); $2xy\ dx + (3x^2 + 4y)\ dy = 0$, $y(1) = 1$.

6) [20pts] Sketch at least two curves in the family $y = cx^3$. Find a formula for the orthogonal trajectories. Sketch at least two of these new curves. It might help to know that $\sqrt{3} \approx 1.7$.

Bonus [5pts]: Show that if $\mu(x,y)$ and $\nu(x,y)$ are integrating factors of the DE

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

and $\mu(x,y)/\nu(x,y)$ is not constant, then

$$\mu(x,y) = c\nu(x,y)$$

is a solution of the DE, for every c.

Remarks and Answers: This exam took approx 60-70 minutes. The average among the top half was approx 80 out of 100, which is very good. The highest score was 102 and there were 5 other grades in the 90's. The averages on each problem ranged from 62% (on #1) to 92% (on #5). The (unofficial) scale for this exam is a little higher than normal:

A's 89-100
B's 79- 88
C's 69-78
D's 59-68

1a) [of Exam A; this is 1b) of Exam B] This has no solution. Plugging in $x = 1$ and $y = 0$ gives $0 + 1 = 0$, a contradiction. The EU Theorem does not apply because $f(x,y) = -x/y$ is not continuous where $y = 0$.

A few people 'solved' this and got $x^2 + y^2 = 1$. This is not a correct solution, but it is an easy mistake to make, so I gave 1-2 points of partial credit.

1b) [of Exam A; this is 1a) of Exam B] We can apply the EU Thm to this. It has ONE solution. Some people actually found it ($ye^{x^2} = 1$) which was OK, but not expected.

1c) This does not fit the EU Thm. It has many solutions, such as $y \equiv 0$ and $y = x^3$. However, it is a bit unusual and we did not cover any examples like it in class (unless Prof Roy did so?), hence the extra credit.

2) TTFFT (but for Exam B, FTTFT)

3) [Exam A] It is separable, and $y(x^2+1)^2 = C$ (or $y = C(x^2+1)^{-2}$).

CHECK: $\frac{dy}{dx} = -4Cx(x^2+1)^{-3} = \frac{-4xy}{(x^2+1)}$, which agrees with the DE.

I also accepted $|y|(x^2+1)^2 = C$ but deducted points for $y(x^2+1)^2 = e^C$ (which does not allow $y \leq 0$). Likewise $|y|(x^2+1)^2 = e^C$ is slightly wrong because it does not include the 'lost' solution $y = 0$.

I did not deduct points if you did not explicitly check your answer, but you should ! I have sometimes taken off points for that in the past, when the problem explicitly asks you to do so.

Exam B is very similar, but leads to $y(x^2+1)^3 = C$.

4) Get $h = 1$, $k = -1$, and convert the DE to $(X - 2Y)\ dX + (2X + Y)\ dY = 0$ which is homogeneous. Set $Y = vX$ and get $\frac{(2+v)\ dv}{v^2+1 = -dX/X}$. Next we'd integrate both sides (except the instructions allow us to stop here), to find v. Use that to find Y, and then $y = Y - 1$. If interested in the final answer, see exercise 2.4.9.

On Exam B, get $h = 3$, $k = 1$. The rest is identical.

5) After the multiplication, the DE is exact, with $M = 2xy^3$. So, $F = \int M\ dx = x^2y^3 + C(y)$. We get $C'(y) = 4y^3$ and $C(y) = y^4$ (or $y^4 + C$). Set $x^2y^3 + y^4 = C$. The I.C. shows $C = 2$ so

$$x^2y^3 + y^4 = 2$$

6) $y' = 3cx^2 = 3(\frac{y}{x^3})x^2 = 3y/x$. Note that this c should be treated with more respect than a typical capital C; so don't replace $3c$ by c, for example. Taking negative recips, we get $\frac{dy}{dx} = -x/3y$ which is separable, and leads to $3y^2 + x^2 = C$ (or some equivalent formula such as $3y^2/2 + x^2/2 = C^2$).

This is a family of ellipses, which most people graphed well enough for full credit. If you set $C = 1$ for example, you get the two points $x = \pm 1$ with $y = 0$ and two points $x = 0$ with $y = \pm 1.7^{-1} \approx \pm.6$. So, these are relatively short and wide ellipses.

Bonus) We should assume that $\mu(x, y) = c\nu(x, y)$ defines y as a function of x and must show that this function solves the DE (think this sentence over a few minutes and compare with the question; the first sentence is often the hardest; almost nobody started this way. In effect, we are going to prove the DE. I will leave just a little of the work below to you.). Taking a differential of μ gives

$$d\mu = \frac{\partial\mu}{\partial x}dx + \frac{\partial\mu}{\partial y}dy$$

with a similar formula for $cd\nu$. Equating them gives

$$\frac{\partial\mu}{\partial x}dx + \frac{\partial\mu}{\partial y}dy = c\frac{\partial\nu}{\partial x}dx + c\frac{\partial\nu}{\partial y}dy$$

and by algebra (*)

$$\frac{\partial[\mu - c\nu]}{\partial x}dx + \frac{\partial[\mu - c\nu]}{\partial y}dy = 0$$

Since μ is an I.F. we know $\mu M dx + \mu N dy = 0$ is exact, and

$$\partial(\mu M)/\partial y = \partial(\mu N)\partial x \quad \text{or}$$

$$\mu\partial M/\partial y + M\partial\mu/\partial y = \mu\partial N\partial x + N\partial\mu\partial x$$

with a similar formula for ν and also for $c\nu$. Subtracting the one for $c\nu$ from the one above, and using $\mu(x,y) = c\nu(x,y)$ to cancel some terms, we get:

$$M\frac{\partial[\mu - c\nu]}{\partial y} = N\frac{\partial[\mu - c\nu]}{\partial x}$$

and comparing this with equation (*) above, we get $M/N = -dy/dx$, which means our function y does solve the original DE. Done.

Remarks on the Bonus: Several people wrote down 'some true stuff' about this situation, but nobody put together a clear and convincing proof. As far as I could tell, nobody set it up right, and tried to prove the DE (many people tried to prove $\mu(x,y) = c\nu(x,y)$ instead, and in many other cases, I just could not see the reasoning or the goal). Usually a proof contains lots of words, maybe 50% words, to put the variables and formulas in context.

This was HW 2.4.17, a relatively hard exercise. Almost nobody asked me about it, and I wondered why not. If you don't get an exercise, ask someone !