MAP 2302 Sept 21, 2015
Exam 1 Prof. S. Hudson

1) [20 pts] a) Show that the DE, $(\frac{dy}{dx})^2 - 4y = 0$, has a one-parameter family of solutions of the form $y = (x + c)^2$.

b) Is this the general solution ? Briefly justify.

c) Does this DE have a unique solution such that $y(1) = 1$? Briefly justify.

2) [20 pts] Find the orthogonal trajectories to this family of curves; $y^2 = cx$.

3) [15 pts] Which of the following are fundamental sets for $y'' + y = 0$? For each list answer yes or no and explain briefly. You should not have to compute any W(x).

a) $\{\sin(x), x\sin(x)\}$

b) $\{\sin(x), \sin(-x)\}$

c) $\{\sin(x), \cos(x + 2)\}$

4) [20 pts] Solve $(x + y)dx - xdy = 0$.

5) [15 pts] Consider $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$. What type of DE is this? Use Ch2 language, such as exact, separable, etc. Show the first step of the solution (for example, set $y =$ something or $\mu =$ something, etc), to show that you know the usual method, but you do not have to solve it.

6) [10 pts] Consider $(5x + 2y + 1)dx + (2x + y + 1)dy = 0$. Show the first step of the solution, but you do not have to solve it.

Bonus) [10 pts] Solve the IVP, $\frac{dy}{dx} + y = f(x)$, where $f(x) = \begin{cases} 5, & \text{if } 0 \le x < 10 \\ 1, & \text{if } x > 10. \end{cases}$
Added 9/24/15: with $y(0) = 6$.

Remarks and Answers: The average score among the top 22 was approx 66 out of 100 (or 110 counting the bonus), with high scores of 99 and 95. The average grade varied from about 45% on problem 1 to 95% on problem 6. The advisory scale, probably more accurate than the one on the syllabus is:

A's 75 to 100
B's 65 to 74
C's 55 to 64
D's 45 to 54

1a) [10pts] $(\frac{dy}{dx})^2 = (2(x + c))^2$ and $4y = 4(x + c)^2$ is equal to that.

This is Ch 1.2.7b, an assigned HW. Don't try to solve this DE (which doesn't fit the EU theorem or any basic Ch2 type), but if you do, watch for ± issues. For example, $\sqrt{y} = x + c$ incorrectly implies $x + c \ge 0$. The results on 1a were better than 1b and 1c.

On problems like 1b and 1c, you have to at least answer Yes or No correctly for any credit. Also, "justify" requires at least a few words.

1b) No. The solution $y = 0$ is not included in the formula $y = (x + c)^2$.

1c) No, it has two, $y = x^2$ and $y = (x - 2)^2$. These are easy to find by solving for c.

2) $y^2/2 + x^2 = k$. Start with $2y\frac{dy}{dx} = c = y^2/x$, so $\frac{dy}{dx} = y/2x$ is the first DE. The second is $\frac{dy}{dx} = -2x/y$, which is separable, with solutions $y^2/2 + x^2 = k$. Be careful with the constants in these problems. Probably $y = \sqrt{cx}$ is a slightly bad start, though a few people got the correct answer that way.

3a) No. $x\sin(x)$ does not satisfy the DE because (shortcut) it does not fit the general solution $c_1\sin(x) + c_2\cos(x)$.

3b) No. Since $\sin(-x) = -\sin(x)$ is a scalar multiple of the first solution, these are LD.

3c) Yes, these are both solutions and are not scalar multiples, so they are LI. [Reminder: this logic does not apply with more than two functions.]

4) $y = x\ln(x) + cx$. This DE is homogeneous (in the Ch2 sense). So, $dy/dx = 1 + y/x$ becomes $v + x\ dv/dx = 1 + v$, $v = \ln(x) + c$, so $y = x\ln(x) + cx$. I strongly suggest checking your answer to such problems. Also, a few people came close to the answer, but their work was indecipherable, so they did not get full credit. If you are not too confused, it only takes a few seconds to write "homogeneous" and organize your work a little.

There are other approaches. The DE is also linear. It also fits a theorem in Ch 2.4 (the one with M_y etc).

5) It is the Bernoulli type with $n = -3$, so set $v = y^{1-n} = y^4$, and you can stop here. I did not consider finding the type to be a step.

6) This is from the Ch 2.4 HW. The first step is to set $x = X + h$ and $y = Y + k$.

It is also exact (which I did not notice when writing the exam). For that, the first step is $F(x, y) = \int 5x + 2y + 1\ dx =$ etc.

B) There was a typo, sorry. Since it is an IVP, it should have an initial condition, such as $y(0) = 6$ (see the corrected version above, which is Ch 2.3.28). Apparently nobody noticed this, though a few people got pretty far into the solution anyway, and I gave them full credit. Since this is linear, we can use $\mu = \exp(\int 1\ dx) = e^x$ and the shortcut $y = e^{-x}\int e^x Q(x)\ dx$. Now there are two possible methods.

1) Set $Q(x) = 5$ and get $y = 5 + Ce^{-x}$ for $0 \le x \le 10$. With the missing IC $y(0) = 6$, we would get $C = 1$ and $y(10) = 5 + e^{-10}$. Use this last equation as an initial condition for a second IVP. Similar work leads to $y = 1 + C_2e^{-x}$ and $C_2 = [4e^{10} + 1]$, for $x > 10$.

Remark: This answer is not differentiable at $x = 10$, but pro's do not find that very disturbing, and have found ways to work with it (eg, the theory of distributions).

2) Or, we could set $Q = f$ though the integration seems a bit awkward, so I prefer method 1.