MAP 2302 | Feb 4, 2016
Exam 1 and Key | Prof. S. Hudson

1) [10 pts] The DE $y'' + y = 0$ has general solution $y(x) = c_1 \sin(x) + c_2 \cos(x)$. Find a solution satisfying the boundary conditions $y(0) = 1$ and $y'(\pi/2) = -1$ (or, if this is not possible, explain).

2) [12 pts] Consider this IVP from Ch 1: $dy/dx = y^{1/3}$ and $y(0) = 0$.

a) Does the EU Thm guarantee a unique solution ? Explain (probably by stating at least part of the theorem, with comments).

b) Show that $y_1(x) = \begin{cases} 0 & \text{if } x \leq 4 \\ [\frac{2}{3}(x-4)]^{3/2} & \text{if } x > 4 \end{cases}$ is a solution.

c) Is the translated function $y_2(x) = y_1(x - 1)$ also a solution ?

3) [15 pts] Find the general solution of $(y^2 + 3)dx + (2xy - 4)dy = 0$.

4) [5 pts] Check that your answer to problem 3 actually solves the DE. You may need implicit differentiation for this. [If you cannot solve (3) you can check your answer to (5) instead, though that may be harder.]

5) [15 pts] Solve the IVP, $(2x - 5y)dx + (4x - y)dy = 0$, $y(1) = 4$.

6) [10 pts] Transform this Bernoulli DE into a linear DE, $\frac{dy}{dx} + y = xy^3$. Write out the new DE in standard form, but do not solve it.

7) [10 pts] State the formula for $\mu(x)$ given in Ch 2.3, used as an integrating factor for $\frac{dy}{dx} + P(x)y = Q(x)$. Then derive (justify) the formula as done in class.

8) [6 pts] In Ex.2.9, we apply $\mu = \frac{1}{(x^2+1)\sin(y)}$ to the separable DE $x \sin(y)\ dx + (x^2 + 1)\cos(y)\ dy = 0$, to get an exact DE. After some work, this family of solutions $\ln(x^2 + 1) + \ln(\sin^2 y) = \ln c$ arises, but it is missing a few "special solutions". Find the special solution with $y(0) = \pi$.

9) [7 pts] Transform $(x + 2y + 3)dx + (2x + 4y - 1)dy = 0$ into a separable DE. Write out the new DE in a standard form, but do not solve it.

10) [10 pts] Find the orthogonal trajectories to the family of curves $xy = c$. If you notice anything strange about this example, you can probably safely ignore it (or you can ask me).

Bonus [5 pts, maybe hard]: Find the general solution of $y'' - 7y' + 10y = 0$, given that there are two solutions of the form $y = e^{kx}$.

Remarks and Answers: The average was approx 63 based on the top half of the scores, which is a bit low. The top six scores were approx 80, ± 1. The average on each problem varied from about 55% to 85% except for problem 2 (23%) and problem 9 (39%). Here is an advisory scale for the exam, based on the relatively low average. It is more accurate than the scale on the syllabus, for now:

A's 75 - 100
B's 62 - 74
C's 52 - 61
D's 42 - 51

If you scored below 40, and do not expect any major changes in study habits, etc, then you should consider dropping the course. If you are having difficulties, but want to stay, see me or Samantha for help.

1) There are many solutions, such as $y(x) = \sin(x) + \cos(x)$, all of the form $c_1 \sin(x) + \cos(x)$. We do not have enough information to solve for c_1. This is exercise 1.3.4b. It illustrates some differences between IVPs (which will be very common this term) and BVPs (less common, and not covered by our EU thms).

A common mistake was to get $c_2 = 1$ but then decide that there are no solutions. Another mistake was to get $c_2 = 1$ and simply stop, with no clear answer. For full credit, you need to answer the question with some formula for $y(x)$. Some people mistakenly calculated a value for c_1 (such as $c_1 = 700$ or whatever, as if it were unique), but I did not deduct for that, if the final answer was OK (such as $y(x) = 700 \sin(x) + \cos(x)$).

2a) No, because $\frac{\partial}{\partial y} y^{1/3}$ is not continuous at 0. It is a good habit to study the main theorems carefully. This example is discussed in section 1.3B and is also exercise 1.3.8 (not assigned).

2b) Check that $y_1'(x) = y_1^{1/3}(x)$ for all x. It is pretty obvious for $x \leq 4$ (0=0). For larger x, $y_1'(x) = [\frac{2}{3}(x-4)]^{1/2} = y_1^{1/3}(x)$ using the Chain Rule. Many people seemed lost on what to do.

2c) Yes. In effect, the translation replaces the three 4's in the formula for y_1 by three 5's, which has no effect on the work in part 2b. There is no significant difference between y_1 and y_2 here. The point of 2b and 2c is that we get *many* solutions to this IVP. This is one way the EU Thm can "fail".

3) Exact. $xy^2 + 3x - 4y = C$.

4) $y^2 + 2xyy' + 3 - 4y' = 0$ leads to $\frac{dy}{dx} = \frac{y^2+3}{4-2xy}$ and the DE of part (3).

5) $(2x+y)^2 = C(y-x)$. The DE is homog (and that should be pretty obvious by now). Set $y = vx$. Get $\int \frac{dx}{x} + \int \frac{4-v}{2-v-v^2} dv = C$ (many people got this far, for approx 11 pts, but few went further). Then use partial fractions to get $\frac{4-v}{2-v-v^2} = \frac{2}{v+2} - \frac{1}{v-1}$. Integrate, etc, to get $(2x+y)^2 = C(y-x)$. Use the IC to get $C = 12$. This problem may seem long, but many DE problems are longer, especially in Ch.6. This is 2.2.19.

6) Sub in $v = y^{1-n} = y^{-2}$ and simplify. Stop at $\frac{dv}{dx} - 2v = -2x$, which is linear. Common mistakes were lack of study (I assume) or to multiply by v. See Ex. 2.17 for the full solution.

7) See the text or lecture notes.

8) FTTF

9) This is a Thm 2.7 case 2 problem. Set $z = x + 2y$ (and $dz = dx + 2dy$) to get $\frac{7}{2}dx + (z - \frac{1}{2})dz = 0$, which is separable.

In general, I did not give partial credit if you did not at least get to $z = x+2y$. There were 2 exceptions: 1) You tried the $x = X + h$ idea from Thm 2.7 case 1 and failed, as you should; or 2) You noticed that the DE is exact (I did not, when writing the exam) and tried to solve it using Ch. 2.1 methods, which unfortunately do not lead to separable DE's, as required. This is Ex.2.20.

10) $y^2 - x^2 = C$. The two DE's are $y' = -y/x$ and $y' = x/y$. It is interesting that the original curves are hyperbolas and the OT's are also hyperbolas. A slightly strange feature is that for each C, we get two separate curves (arguably, two solutions) rather than a single curve, but this did not seem to bother anyone.

B) $y(x) = \pi$, see Ex 2.9 for more on this.