MAP 2302 July 19, 2013
Exam 2 and Key Prof. S. Hudson

Due to computer issues, some wordings below may vary slightly from the actual exam.

1) [10 pts] Given that x, x^2 and x^4 are all solutions of

$$x^3y''' - 4x^2y'' + 8xy' - 8y = 0$$

show that they are LI on $0 < x < \infty$ (including a brief explanation).

2) [15 pts] 2a) Find the general solution of $y'' + 2y' + 5y = 0$. Show all your work up to finding the roots of a polynomial, but then you can simply write the answer, if you want.

2b) [with help from part (a)] Given that $y = x^2$ satisfies $y'' + 2y' + 5y = 5x^2 + 4x + 3$ and that $y = \sin(x) + 5e^x \sin(2x)$ satisfies $y'' + 2y' + 5y = 2\cos(x) + 4\sin(x)$, find the general solution of

$$y'' + 2y' + 5y = 5x^2 + 4x + 3 + 2\cos(x) + 4\sin(x)$$

3) [20 pts] Answer True or False. You do not have to explain, unless you think some part is ambiguous.

The UC set of $x^n e^x$ contains exactly n functions.

If f_1 and f_2 solve the usual Ch 4.6 DE (see problem 7), and $f_1(x_0) = f_2(x_0) = 0$, then the two solutions are L.D.

If a motion satisfies $x'' + 4x' + 16x = 0$, then it is under-damped and oscillatory.

The points $x = 0$ and $x = 1$ are ordinary points of the *Legendre* equation $(1 - x^2)y'' - 2xy' + 12y = 0$.

The Wronskian of x^3 and x^4 is nonzero for all x.

4) [15 pts] Use the method of under-determined coefficients to solve the IVP, $y'' - 4y = 16xe^{2x}$, $y(0) = 4$, $y'(0) = 8$. For about 5 points EC, check that your answer solves the IVP (showing all work).

5) [10 pts] An 8-lb weight stretches a hanging spring 6in from its natural position. The weight is then pulled down another 6in and released at time $t = 0$. The medium offers resistance equal to $4x'$ where x' is the velocity in feet per sec. Find a formula for the displacement $x(t)$. Note that gravity is $g = 32$ ft/sec^2.

6) [15 pts] There are two LI series solutions centered at $x_0 = 0$ to Legendre's equation (with $n = 1$),

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

6a) Show that the recursion formula for the coefficients is $c_{k+2} = \frac{c_k(k-1)}{k+1}$.

6b) Find two LI series solutions (one should be a simple polynomial).

7) [15 pts] Choose one proof. You can assume that f_1 and f_2 solve the usual DE from 4.6, $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, with continuous coeffs and $a_0(x) \neq 0$.

a) If the Wronskian of f_1 and f_2 is zero for all x, then they are L.D.

b) If f_1 and f_2 are L.I., then every other solution is a LC of them.

Bonus [5pts]: In exercise 1, pg 132 of the text, the author claims that the general solution of

$$x^2y'' - 4xy' + 4y = 0$$

is $c_1x + c_2x^4$. Show that this is false by finding a solution on the interval $(-1, 1)$ that does not take this form. Hint: consider absolute values.

Remarks and Answers: The average was 67, with high scores of 92, 90 and 85, which is pretty normal. The best results were on problem 1 (86%). The worst results were on problem 4 (46%). The exam was intended to be an hour, but actually went about 1.5 hours. The rough scale for Exam 2 is:

A's 75 - 100
B's 65 - 74
C's 55 - 64
D's 45 - 54

I have also computed your semester average based on the two quizzes and two exams. It appears on the upper right corner of your Exam 2 and represents 50% of your semester grade (the rest will come from the final and the HW). The average of these averages among the top 25 students was 70, with highs of 93 and 92. I estimated your current letter grade using this (still unofficial) scale:

A's 78 - 100
B's 68 - 77
C's 58 - 67
D's 48 - 57

1) Compute $W(x) = 6x^4$. So, for example, $W(1) = 6 \neq 0$. A theorem says the functions must be LI, since the Wronskian is nonzero at some point in $(0, \infty)$.

2a) $y_c = e^{-x}(c_1 \sin(2x) + c_2 \cos(2x))$ from the QF, which gives $m = -1 \pm 2i$.

2b) $y = y_c + y_{p_1} + y_{p_2} = e^{-x}(c_1 \sin(2x) + c_2 \cos(2x)) + x^2 + \sin(x)$. Do not include $5e^{-x}\sin(2x)$, which is included in $e^{-x}(c_1 \sin(2x)$. We did examples like this in class - or see page 131. This was supposed to be easy, but several people tried the UC method, which takes much longer, risks errors, and ignores the given info.

3) FTTFF Here are brief explanations:

· You can set $n = 0$ to see the claim fails. Or you can write out the UC set (see Table 4.1), and see it has $n + 1$ functions.

· See the text.

· Use the Q.F. The complex roots imply a trig function in the answer. See pg 201 for the vocab.

· 1 is a root of $1 - x^2$, so it is a sing. pt.

· $W(x) = x^6$ is zero when $x = 0$. You might not need to actually compute W to guess this happens.

4) $y_c = c_1 e^{2x} + c_2 e^{-2x}$ so the UC set must be changed from $\{e^{2x},\ xe^{2x}\}$ to $\{xe^{2x},\ x^2 e^{2x}\}$. Then $y_p = Ax^2 e^{2x} + Bxe^{2x}$ leads to $y_p = 2x^2 e^{2x} - xe^{2x}$. So, $y(x) = c_1 e^{2x} + c_2 e^{-2x} + 2x^2 e^{2x} - xe^{2x}$ and the initial conditions lead to $y(x) = \frac{17}{4} e^{2x} - \frac{1}{4} e^{-2x} + 2x^2 e^{2x} - xe^{2x}$.
Note that you should not use the I.C's until you have included y_p.

5) The DE is $x'' + 16x' + 64x = 0$ so $x(t) = c_1 e^{-8t} + c_2 te^{-8t} = \frac{1}{2} e^{-8t} + 4te^{-8t}$. This is *critically damped* motion.

6) Part a) is similar to many examples in the text. For b), we must leave c_0 and c_1 arbitrary. Set $k = 0$ to get $c_2 = -c_0$. Set $k = 1$ to get $c_3 = 0$, etc. You should see a clear pattern, that the even terms are all multiples of c_0 and lead to $y(x) = c_0[1 - x^2 - x^4/3 + \ldots]$. There is only one nonzero odd term, leading to just $y(x) = c_1 x$ (the polynomial).

7) See the text or lecture notes for the proofs. Note that proofs are our main tool in creating new mathematics, as important as experiments in the sciences. Here, you were expected to study and understand the announced textbook proofs, and learn to write them down, possibly in your own words. Generally such a proof needs at least as many sentences as formulas, to guide the reader through the logic of the proof. That was the main problem I saw. Many of the answers were practically impossible to follow and I could not give much credit for those.

Bonus) Let $y(x) = |x|x^3$, which is the same as x^4 for $x \geq 0$, but is $-x^4$ for $x < 0$. You can check that it is a solution, even at $x = 0$, and is not a $c_1 x + c_2 x^4$. This is similar to a remark I made in class, but it is probably a bit difficult, and nobody got it.

I advised you not to work on bonus questions like this until you have finished the other problems and checked them. About 60% of the class tried this one, which indicates that approximately enough time was given for this exam.