MAP 2302 Oct 16, 2014
Exam 2 and Key Prof. S. Hudson

1) [10 pts] Find the orthogonal trajectories to the family of curves $cx^2 + y^2 = 1$.

2) [15 pts] Answer True or False. You do not have to explain, unless you think some part is ambiguous.

The Wronskian of $x\sin(x)$ and $x\cos(x)$ is nonzero for all x.

The Wronskian of e^x and e^{2x} is nonzero for all x.

The UC set of $x^2\sin(x)$ contains exactly 6 functions.

If a free undamped motion satisfies $x(t) = 3\sin(5t) + 4\cos(5t)$ then it oscillates with an amplitude exceeding 4.32101.

If a free damped motion satisfies $x'' + 2x' + 8x = 0$, then it is under-damped and oscillatory.

3) [15 pts] Find the general solution of $y'' + y = \sec^3(x)$ by variation of parameters. I'd suggest the shortcut ($v_1 = -\int \frac{F\ y_2}{a_0 W}$ etc).

4a) [15 pts total] Find the general solution $y(t)$ of $y'' - 8y' + 16y = 0$.

4b) Find the general solution of $y'' - 8y' + 16y = 3e^{2t}$.

4c) Write the general form for a particular solution of $y'' - 8y' + 16y = 5t\sin(t) + 3e^{4t} + 10$ including constants A, B, C etc as needed (using the UC method). You do not have to compute the constants.

5) [15 pts] Given that $y_1(x) = x$ is a solution of $(x^2 - 1)y'' - 2xy' + 2y = 0$, find the general solution. Small hint: the formula $\frac{-2}{x} + \frac{1}{x+1} + \frac{1}{x-1}$ may arise, about halfway through.

6) [15 pts] Find the general solution of $9x^2y'' + 3xy' + y = 0$ for $x > 0$.

7) [15 pts] Choose one proof. You can assume f_1 and f_2 solve the usual DE from 4.6, $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, with continuous coefficients, with $x > 0$ and $a_0(x) > 0$. As usual, LI and LD abbreviate *linearly (in)dependent*.

a) If $f_1(3) = f_2(3) = 0$ then f_1 and f_2 are L.D.

b) If f_1 and f_2 are LI, then every other solution is a LC of them.

Bonus [5pts]: In class, I gave a very rough proof of Thm 4.6.19, that $W(x)$ is either always zero, or never zero (not expecting many people to remember it well). Repeat as much of that as you can remember, including at least the new form for W, and some explanation.

Remarks and Answers: The average was 76 out of 100, based on the top half, which is down a little from Exam I, but still very good. The high scores were 95 and 91. The results were generally good on all the problems except the Bonus and perhaps the proof (about 55%). Please see me about your Exam I score, or see the HW page. The unofficial scale for Exam II is

A's 82-100
B's 72-81
C's 62-71
D's 52-61

1) $x^2 + y^2 - 2\ln y = K$. Some of the key steps are $2cx + 2yy' = 0$, $c = (1-y^2)/x^2$ and $y' = -(1-y^2)/xy$. The new DE is $y' = xy/(1-y^2)$, which is separable and leads quickly to the answer.

2) F T T T T

3) $y = y_c + y_p$ where $y_c = c_1\sin(x) + c_2\cos(x)$ (most people got this part right just from memory, which was OK). Also, $y_p = v_1(x)\sin(x) + v_2\cos(x) = \tan(x)\sin(x) - \frac{1}{2}\sec(x)$. This comes from $\int \sec^2 = \tan$ and from $-\int \frac{\sin(x)}{\cos^3(x)} = -\frac{\sec^2(x)}{2}$ (using $u = \cos(x)$). It is also OK to solve the last integral using $-\int \tan(x)\sec^2(x) = -\tan^2(x)/2$ (using $u = \tan(x)$), which leads to an equivalent answer (though it may look different).

A few people got y_c wrong, which made the continuation (and partial credit) pretty difficult. I'd suggest special care at the beginning of any moderately long problem like this.

4a) $y = c_1e^{4t} + c_2te^{4t}$

4b) $y = c_1e^{4t} + c_2te^{4t} + (3/4)e^{2t}$.

4c) $y_p = At^2e^{4t} + B\sin(t) + Ct\sin(t) + D\cos(t) + Et\cos(t) + F$.

Grading notes: On 4a), I did not deduct points for using x instead of t, but if you continued to mix-up the letters through 4c, I deducted a point. On 4b), I gave credit for $y_c + (3/4)e^{2t}$ even if you had y_c wrong in 4a). On 4c), you should not include y_c, since you are only setting up a UC calculation, though I sometimes let that go, if everything else was perfect.

5) Reduction of Order: $c_1x + c_2(x^2+1)$. The usual method leads to $w'/w =$ the partial fractions given in the hint, then to $w = (x+1)(x-1)/x^2$, etc. It is also OK to memorize and use the shortcut formula for v (though a surprising number of people got stuck doing this).

6) Cauchy-Euler. $c_1x^{1/3} + c_2x^{1/3}\ln(x)$. Some people memorized the CE pattern and jumped to $9y'' - 6y' + y = 0$ (in y, t), which I accepted. But if there were mistakes, I gave more partial credit to people who worked it all out, starting with $x = e^t$.

7) For Part b), see the text or lecture notes. Some answers were rambling essays that did

not include the main ideas, such as 1) using $W(x_0)$ theorems to get the c_1, c_2 needed to define g and 2) using the EU theorem to show $f = g$ which is a LC by definition.

Part a) is shorter and is a slight variation of a HW problem. Many people got the main idea, but didn't explain it well, and/or included too much irrelevant stuff (such as the definition of LD). Here is a short clean proof:

Proof of b): By the definitions of W and the determinant, $W(3) = f_1(3)f_2'(3) - f_2(3)f_1'(3)$. Since $f_1(3) = f_2(3) = 0$, this simplifies to $W(3) = 0$. By a theorem in Ch 4.6, the f_j are LD.

Bonus) The key formula is $W(x) = ce^{B(x)}$ (for full credit explain briefly what B is, from the DE for W). This is zero for one x if only if $c = 0$, so that $W = 0$ for all x.

Remark: according to my notes, most of the exam problems came from these textbook exercises, maybe with small changes:

3.1.3
4.4.2
4.1D.3
4.5.9
4.6.3