Exam III | Nov 13, 2014
MAP 2302 | S Hudson

1) [15pts] Begin using the Method of Frobenius to find two LI solutions to $x^2y'' - xy' - (x^2 + \frac{5}{4})y = 0$ on some interval $0 < x < R$. But you do NOT have to find the y_j. You can stop after finding the root(s) of the indicial equation. Show your work, including several series as usual, and the formula relating c_k to other coefficients and/or variables.

2) [20 pts] True-False.

The DE $(x-1)y'' + xy' + \frac{y}{x} = 0$ has exactly two singular points.

The DE $x^2(x-2)^2y'' + 2(x-2)y' + (x+1)y = 0$ has two irregular singular points.

The function e^x is analytic at 0.

$\Gamma(5) > 50$.

The function e^{t^2} has exponential order.

3) [15 pts] A 32-lb weight stretches a hanging spring 2 ft. from its natural position. The weight is then pulled down another 6in and released at time $t = 0$. The medium offers resistance equal to $8x'$ where x' is the velocity in feet per sec. Find a formula for the displacement $x(t)$ and classify the motion as under-damped, critcally damped, or overdamped. Note that gravity is $g = 32$ ft/sec^2.

4) [15 pts] Let $f(t) = t$ for $t > 0$. Use the definition of the Laplace transform to find $L(f)$. Also, state its domain, with a brief explanation.

5) [5 pts] Give an example like problem 3) that leads to simple harmonic motion. You may simply list some changes to problem 3) if you like. Write out the revised DE and one non-zero solution.

6) [15pts] Find a power series solution centered at $x_0 = 1$ for $y'(x) = 2(x-1)y$, with $y(1) = 3$. Include at least the first three nonzero terms.

7) [15pts] The function $J_0(x)$ below solves Bessel's equation of order 0, $x^2y''+xy'+x^2y = 0$ on some interval $0 < x < R$. Recall that the indicial equation is $r^2 = 0$. Some of the other work behind J_0 is also given below, though you may not need it. Apply Reduction of Order to find a second L.I. series solution $y_2(x)$. Include the $\ln(x)$ term, if any, and at least two other nonzero terms. You are given:

$(n + r)^2c_n + c_{n-2} = 0$, for $n \geq 2$ and

$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \ldots$.

Bonus: State Bessel's equation of order 1/2 and give two *explicit* L.I. solutions in closed form. (eg, understandable to a Calculus I student - without the J or Y notation, etc, and not in series form).

Remarks: The average was approx 69 out of 100, based on the top 18 scores out 28, with highs of 91 and 83. The average scores were highest on Problem 1 (90%) generally decreasing from Problem 2 to Problem 7 (55%). In problems 3 and 6, a bad start could change the problem completely, so that little partial credit was possible on those. Remember to read each problem carefully, start it carefully, and check afterwards that you have answered all parts. Mistakes near the end of a problem should be avoided too, of course, but those usually have less effect on your grade. Scale:

A's 77 - 100
B's 67 - 76
C's 57 - 66
D's 47 - 56

I wrote your semester exam average in the upper right corner of your Exam III (check). The class average for that seems to be about 71, two points above Exam III. This number is a bit uncertain until I can fix all the Exam I scores, a nameless exam, etc, and it could eventually rise to approx 75. But assuming it is correct, you can adjust the scale above by 2 points (with A-'s starting at 79, etc). I will include HW, ±'s and EC later.

Answers:

1) This is Ex 6.13 from page 261; $r_1 = 5/2$ and $r_2 = -1/2$. For the 'formula' part, equation (6.80) would be ideal, but I usually accepted equations like (6.78), preferably without the $\sum$.

2) TFTFF

3) The IVP is $x''(t) + 8x'(t) + 16x(t) = 0$ with $x(0) = 0.5$ and $x'(0) = 0$. The solution is $x(t) = 0.5e^{-4t} + 2te^{-4t}$. This is critically damped motion. See also Ex. 5.2, which this was based on.

4) $L(f) = \int_0^\infty e^{-st}t\ dt = \frac{1}{s^2}$ for $s > 0$. Explanation of the domain: the calculation uses $\lim_{M\to\infty} e^{-sM} = 0$, which is only true for $s > 0$.

It's best to include the dt and the $\lim_{M\to\infty}$ in these calculations to avoid various careless errors, and this might also remind you of the domain issue.

5) In problem 3, if the resistance is zero, we get $x''(t) + 16x(t) = 0$ with solutions such as $x = \sin(4t)$.

6) $3 + 3(x-1) + \frac{3}{2}(x-1)^2 + \ldots$. Start with $y = \sum_0^\infty c_n(x-1)^n$ and go through the usual routine. Set $c_0 = y(1) = 3$. After handling the non-zero center, this should be slightly easier than normal, since this is a first order DE. It is OK to start with $t = x - 1$ and then set $y = \sum_0^\infty c_n t^n$. It is also possible to 'cheat' a bit by solving with Ch.2 methods, to get $y = 3e^{(x-1)^2}$, and then replace that function by its Taylor Series (= the answer above).

Starting with $y = \sum_0^\infty c_n x^n$ was a pretty bad and common mistake. Maybe people misread the problem. But it is hard to find the series solution centered at 0, mainly because the initial condition is at x=1. I *think* the answer to that should look like

$$3e(1 - 2x + 3x^2 \ldots)$$

If you got this incorrect answer, I gave roughly 8 points partial credit, but not much for others.

7) This is on pages 272-273, with final answer $y_2(x) = J_0(x)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} \ldots$

B) $x^{-1/2}\sin(x)$ and $x^{-1/2}\cos(x)$. See Ch.6.3.3. I did not give partial credit without some formula(s) at least close to these.