MAP 2302 — Aug 2, 2013

Final Exam — Prof. S. Hudson

NAME

Show all your work and reasoning for maximum credit. If you continue your work on the back/etc, leave me a note. Do not use a calculator, book, or any personal paper. If you need extra paper, ask me, and then hand it in with your exam.

1) [8pts] Find the general solution, $y' = \frac{y}{x} - \frac{y^2}{x}$.

2) [10pts] Solve the Cauchy-Euler I.V.P for $x > 0$: $x^2y'' - 2y = 8$, $y(1) = 0$, $y'(1) = 5$. You may be able to guess a very simple y_p.

3) [6pts] Solve the I.V.P. $(2xy - 3)dx + (x^2 + 4y)dy = 0, \quad y(1) = 2$.

4) [6pts] Use the 'UC' method to find a particular solution of the *first order* DE: $y' - y = x\sin x$. Hints: the method is the same as for second order DEs, $y_c = ce^x$, and the UC set of $x\sin x$ is $\{x\sin x, x\cos x, \sin x, \cos x\}$.

5) [5pts] Are you most confident of your answer to (2), (3) or (4) above ? Check that answer carefully below, showing all work.

6) [8pts] Find a series solution to the I.V.P: $y'' = 2xy$, $y(0) = 1$, $y'(0) = 0$. Find the first 3 nonzero terms.

7) [10pts] Answer True or False. You do not have to explain, unless you think the problem is ambiguous.

In class, we discussed Leibniz and Euler more than Newton and Cauchy.

If f and g are L.I. solutions of a Cauchy-Euler equation on $(-\infty, \infty)$, then $W(f, g)(x) \neq 0$ for all x.

Solutions in Ch.5.2 (free undamped motion) often take the form $x = c\tan(\sqrt{\frac{k}{m}}t + \phi)$.

The only solution to $y' = x^2y^2$ with $y(0) = 0$ is the trivial one, $y(x) \equiv 0$.

One fundamental set of solutions of $y^{(4)} - y = 0$ (a 4th order linear DE) is $\{\sin x, \cos x\}$.

8) [7pts] Ex.6.14 uses the method of Frobenius on $x^2y'' + (x^2 - 3x)y' + 3y = 0$. The indicial equation is $r^2 - 4r + 3 = 0$. The first solution is a power series that simplifies to $y_1(x) = x^3e^{-x}$. Note that you can *easily* write y_1 and $\frac{1}{y_1}$ as series, if needed. Use reduction of order to find a second LI solution $y_2(x)$, which is a series plus a term with $\ln(x)$. To save time, you only have to find the *first* term of the series part (and then the $\ln x$ part).

9) [5pts] Compute the first 3 terms of the series for $[1 + 3x + 5x^2 + 7x^3 + \cdots]^{-2}$.

10) [8pts] Choose ONE proof (these require relatively little explanation, but try to justify most of the steps). For maximum credit, avoid bad notation such as $f(x)|_0^\infty$ in your work.

a) $L(f') = sL(f) - f(0)$ (assume f is nice, and s is large enough)

b) $\Gamma(N+1) = N\Gamma(N)$ for $N > 0$ (not necessarily an integer).

11) [7pts] Start solving this system using the Laplace transform. You can stop when you have an explicit formula for $L(x)$.

$x' - 4x + 2y = 2t$

$y' - 8x + 4y = 1$

$x(0) = 3$, $y(0) = 5$

12) [5pts] Find $L(f)$ using the u_a method (other methods get partial credit up to 4 points).

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 3 \\ 0 & \text{if } 3 < t < 6 \\ 2 & \text{if } t > 6 \end{cases}$$

13) [5pts] Find $L(2 + \delta(t-3))$.

14) [5pts] Find $L^{-1}(F)$, if $F(s) = \frac{6s}{(s^2+9)^2}$. This assumes you know the Table. If not, convolution or integrating $F(s)$ might help.

15) [5pts] Find $L^{-1}(F)$, if $F(s) = \frac{10s}{(s+3)(s^2+1)}$. Any valid method is OK, but PF's are suggested.

Bonus) [5pts] Given the matrix hints below, find the general solution $(x(t), y(t))$ of the system, $x' = x + y$, $y' = 2x$.

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Remarks and Answers: The average was approx 62, with high scores of 90 and 84. The best results were on #3 (about 90%), while the worst were on #8 (about 35%), then #5 and #9 (about 45%). I do not usually set an explicit scale for finals, though the semester grades will be scaled.

1) $|\frac{y}{y-1}| = C|x|$, though there are several equivalent forms of this answer, and I considered the absolute values optional. Since this is Bernoulli, you can set $v = y^{-1}$. It is also separable. A few people set $y = vx$ (though the DE is not homogeneous) and got the right answer. Many people made careless errors.

2) You can set $x = e^t$ as in the text, or $y = x^r$ instead, getting $y_c = c_1x^2 + c_2x^{-1}$. You can probably guess $y_p = -4$ (if not, use UC). The IC's give $y = 3x^2 + x^{-1} - 4$ (note that $y(1) = c_1 + c_2 - 4 = 0$).

The most common mistakes were about organization (such as including the 8 while looking for y_c, or forgetting to use the 8 at all, or using the IC's before finding y_p).

3) It is exact, so $F(x, y) = \int 2xy - 3\ dx = x^2y - 3x + C(y)$ with $C(y) = 2y^2$. Ans: $x^2y - 3x + 2y^2 = 7$ (do not include the letter F in your answer).

4) Starting with $y = Ax\cos x + Bx\sin x + C\cos x + D\sin x$, the DE gives a 4x4 system. Get $y = -\frac{1}{2}x\cos x - \frac{1}{2}x\sin x - \frac{1}{2}\cos x$.

Do not include $+ce^x$ into the final answer, since the problem asks only for a particular solution. The formula for y_c is not used, except to check that we don't need to adjust the UC set.

5) I will check Problem (2); $y = 3x^2 + x^{-1} - 4$ and $y' = 6x - x^{-2}$ and $y'' = 6 + 2x^{-3}$. Then $D(y) = 6x^2 + 2x^{-1} - 2[3x^2 + x^{-1} - 4] = 8$. And $y(1) = 3 + 1 - 4 = 0$ and $y'(1) = 6 - 1 = 5$ so it all checks.

6) $1 + \frac{x^3}{3} + \frac{x^6}{45} + \cdots$. The recurrence relation is $c_{n+2} = \frac{2c_{n-1}}{(n+1)(n+2)}$.

7) TFFTF Note that the Wronskian theorem does not apply to CE eqns if the domain includes $x = 0$.

8) $y_2 = \frac{1}{2}x^3 e^{-x}\ln x + [-x/2 + \cdots]$ Start with $y_2 = y_1 \int \exp(-\int(1 - 3/x)) \cdot y_1^{-2}\ dx = y_1 \int x^{-3}e^x$. Write $x^{-3}e^x = x^{-3} + x^{-2} + x^{-1}/2 + \cdots$ as a series, integrate it, then multiply by $y_1 \approx x^3$.

9) $1 - 6x + 17x^2 \cdots$. I'd start with squaring, to get $1 + 6x + 19x^2 \cdots$ and then divide that into 1. Or you can invert first, $1 - 3x + 4x^2 \cdots$ and then square. These are Calc 2 skills used in Ch 6.2-6.3.

10) See the text or lecture notes. Almost everyone chose the advertised part A.

11) $X(s) = \frac{3s^3 + 2s^2 + 8}{s^4}$ and you can stop there.

12) $\frac{2}{s}[1 - e^{-3s} + e^{-6s}]$

13) $2/s + e^{-3s}$

14) $t\sin(3t)$

15) $-3e^{-3t} + 3\cos(t) + \sin(t)$

B) One is $x(t) = e^{2t},\ \ y(t) = e^{2t}$. Another is $x(t) = e^{-t},\ \ y(t) = -2e^{-t}$. Then take LC.