MAP 2302 | Dec 11, 2014
Final Exam | Prof. S. Hudson

1) Solve the IVP: $xy' - 2y = 2x^4$ with $y(2) = 8$.

2) Find the general solution, $y''' + 4y'' + y' - 6y = -18x^2 + 1$.

3) Use a Wronskian to decide whether these two solutions of $y'' + 4y = 0$ are L.I. Mention which aspects of the DE are important to your reasoning (the order ? linearity ? etc ?).

$y_1 = 4\cos(2x)$
$y_2 = 1 - 2\sin^2(x)$

4) [typo corrected] Write the general solution to the DE (Ex 6.12), $2x^2y''+xy'+(x^2-3)y = 0$ using the work below. You should not actually have to use any $\sum$'s. Just finish the work, including a couple of new nonzero terms.
$r_1 = 3/2$ and $r_2 = -1$
$y_1(x) = x^{3/2}(1 - \frac{x^2}{18} + \ldots)$
For r_2, c_0 is unknown (as usual), but $c_1 = 0$.
The recursion formula is $n(2n-5)c_n + c_{n-2} = 0$ (for $n \geq 2$). Finish this!

5) Use a convolution to compute the inverse Laplace transform, $L^{-1}(\frac{1}{s^2+5s+6})$.

6) Find $L(f)$ using the u_a method, where

$$f(t) = \begin{cases} 1 & \text{if } 0 < t < 2 \\ 2 & \text{if } 2 < t < 4 \\ 3 & \text{if } 4 < t < 6 \\ 0 & \text{if } t > 6 \end{cases}$$

7) [20pts] Answer True or False. You do not have to explain.

$L(t\cos(bt)) = \frac{2bs}{(s^2+b^2)^2}$.

$L(u_5(t)) = \frac{e^{-5s}}{s}$.

$t^2\sin(5t)e^{3t}$ is of exponential order.

$2\Gamma(3/2) = \Gamma(1/2)$.

$J_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(n+1)!}$

$x^2y'' + 5y' - xy = 0$ has an irregular singular point at $x = 0$.

A free over-damped motion can pass through the equilibrium position multiple times.

Solving a Cauchy-Euler DE, the usual transformation is $v = y^{1-n}$.

The basic EU Thm (in Ch.1) assumes that $f(x, y)$ is continuous.

The DE $(2xy + 1)dx + (x^2 + 4x)dy = 0$ is exact.

8) Use a Laplace transform to solve this IVP, where δ is the usual Dirac mass: $y''+4y'+3y = \delta(t - \pi)$ with $y(0) = 1$ and $y'(0) = -3$.

9) Solve this system: $2x' + y' - x - y = e^t$ and $x' + y' + 2x + y = e^t$.

You are also given that $y(0) = 1$ and that $x(t) = 8\sin(t) + 2\cos(t)$. This much information is unusual, and it should allow you to solve the system without the Laplace transform, but you can use that if you want.

Remarks and Answers: The average was 72 out of 100, based on the top 18 students. The highest scores were 98 and 84. The average score on each problem did not vary much, from approx 60% (on #1) to 80% (on #4,5). I do not set a separate scale for the final, and have not yet set a scale for the semester grade.

1) $y = x^4 - 2x^2$. See 2.3.19. Divide by x first, then get $\mu = x^{-2}$. If you have forgotten Ch.2, you can probably (but I have not written this out carefully) use Ch. 4 methods and get $y_c = Cx^2$ and $y_p = x^4$ and then $C = -2$, with the same answer.

2) $y = c_1 e^x + c_2 e^{-2x} + c_3 e^{-3x} + 3x^2 + x + 4$. See 4.3.15.

3) $W = 0$, so they are LD. This reasoning uses a theorem about *linear homogeneous* DE's (which you should mention). The calculation and simplification of W is easier if you notice from the start that $y_2 = \cos(2x)$. Otherwise, the formula for W is a bit long, but you can set $x = 0$ (for example) to see quickly that $W = 0$. People made many different mistakes on this problem.

4) $y = C_1 y_1 + C_2 y_2 = C_1 x^{3/2}(1 - \frac{x^2}{18} + \ldots) + C_2 x^{-1}(1 + \frac{x^2}{2} + \ldots)$. In the grading, I was mostly interested in seeing the $1 + \frac{x^2}{2}$ part, but many people forgot to write the general solution, including the C_j (or some similar notation).

5) $FG = \frac{1}{(s+2)(s+3)}$, $f * g(t) = \int_0^t f(x)g(t-x)\, dx = \int_0^t e^{-2x} e^{-3(t-x)}\, dx = e^{-2t} - e^{-3t}$.

Plan B: As mentioned in class, I usually prefer PFs on such problems. But on this exam I wanted to test you on convolution and I required that method for full credit. If you used PFs here, you got 7 points (max).

Plan C: If you used PFs first, to get the answer, and then faked some knowledge of convolution, forcing it to give the same answer despite several gaps or mistakes, probably trying to trick the grader, you got 6 points (max).

6) $L(1 + u_2 + u_4 - 3u_6) = (1 + e^{-2s} + e^{-4s} - 3e^{-6s})/s$

7) FTTTF TFFTF

8) Start with $Y = \frac{e^{-\pi s}+s+1}{(s+1)(s+3)} = e^{-\pi s}[\frac{1}{2(s+1)} - \frac{1}{2(s+3)}] + \frac{1}{(s+3)}$, from a PF calculation. So, $y(t) = e^{-3t} + g(t-\pi)$, where $g(x) = (e^{-x} - e^{-3x})/2$ if $x > 0$ (and g=0, otherwise).

This is ex.9.4D.5; see the textbook answer key for an equivalent answer that does not use the letter g. Few people made it through this problem with no silly errors.

9) This was supposed to be ex 9.5.7 with half the answer provided, but there is a typo at the end of the first DE. So, the formula for x is incorrect, but as long as you used it in a logical way, I gave credit. There are several plans.

Plan A: Ignore the formula for $x(t)$ except to get an IC, $x(0) = 2$. Then solve the system with the Laplace method:

$$(2s-1)X + (s-1)Y = \frac{5s-4}{s-1}$$

$$(s+2)X + (s+1)Y = \frac{3s-2}{s-1}$$

Combine to remove X, solve for Y, then use PFs, and the inverse transform:

$$Y = \frac{s^2+s+10}{(s-1)(s^2+1)} = \frac{6}{s-1} + \frac{-5s-4}{s^2+1}$$

$$y = 6e^t - 5\cos(t) - 4\sin(t)$$

Plan B: $x'(t) = 8\cos(t) - 2\sin(t)$. Sub this and x into the first DE (ignoring the 2nd):

$$2[8\cos(t) - 2\sin(t)] + y' - [8\sin(t) + 2\cos(t)] - y = e^t$$

$$y' - y = e^t - 14\cos(t) + 12\sin(t)$$

This is now roughly a Ch.2 or Ch.4 problem; $y_c = Ce^t$ and $y_p = te^t - 13\sin(t) + \cos(t)$ (using UCs). Adding (and noting that the IC implies $C = 0$):

$$y(t) = te^t - 13\sin(t) + \cos(t).$$

The solutions from Plans A and B do not match because of the typo. Other plans (involving various algebra tricks, for example) are possible, which can lead to other answers worth full credit.