MAP 2302 Dec 11, 2015
Final Exam and Key Prof. S. Hudson

1) Find an integrating factor for the DE $\frac{dy}{dx} + 3y = 3x^2e^{-3x}$, then transform it into an exact equation and confirm that the resulting DE is exact. You do not have to solve it.

2) Find an equation for the orthogonal trajectories to this family of curves: $x^2 + y^2 = cx^3$.

3) As you probably know, the DE $y'' + y = \tan(x)$ has the complementary solution $y_c(x) = c_1 \sin(x) + c_2 \cos(x)$. Find a particular solution $y_p(x)$ using one of the methods from Ch.4.

4) Begin the method of Frobenius on the DE, $x^2y'' - xy' + (x^2 + \frac{8}{9})y = 0$. Stop when you have found r_1 and r_2. Note: this is from Ch.6.2 and is *not* a Bessel equation.

5) Solve the IVP, $y'' + y = \delta(t - \pi)$, $y(0) = 0$, $y'(0) = 1$, where δ is the Dirac function.

6) Compute the Laplace transform of $\sin(2t)$ using the definition of the transform (or you can use a certain IVP, as done in class, if you prefer). For full credit, do not simply write the answer from memory.

7) [6 pts] Given that $m^4 - 4m^3 + 14m^2 - 20m + 25 = ((m - 1)^2 + 4)^2$, find the general solution of $y^{iv} - 4y''' + 14y'' - 20y' + 25y = 0$.

8) Compute the inverse Laplace transform of this function. Hint: one of the partial fractions is $\frac{s-3}{s^2+1}$.

$$\frac{5s^2 - s - 2}{(s + 2)(s^2 + 1)}$$

9) Compute the convolution of $\sin(3t)$ with itself. Hint: you may need the identity

$$\sin(A)\sin(B) = (\cos(A - B) - \cos(A + B))/2.$$

10) [6 pts] State the formula for the transform of a periodic function f, such that $f(t+P) = f(t)$ for all $t > 0$.

11) [16 pts] Answer True or False. You do not have to explain.

$L^{-1}(\frac{2bs}{(s^2+b^2)^2}) = t\cos(bt)$.

$L(u_4(t)) = e^{-4s}$.

The indicial equation of $x^2y'' + xy' + (x^2 - 4)y = 0$ is $r^2 - 4 = 0$.

$x^2y'' + 5y' - xy = 0$ has a regular singular point at $x = 0$.

A free underdamped motion can pass through the equilibrium position multiple times.

A free underdamped motion can be a strictly decreasing function of t on $[0, +\infty)$.

Solving a Bernoulli DE, the usual transformation is $v = y^{1-n}$.

The basic E.U. Thm. in Ch.1 applies to the IVP $y' = x^2 \sin(y)$ with $x_0 = 1$ and $y(1) = -2$.

12) Choose ONE. As usual, include enough words.

a) State and prove the formula for $\Gamma(N+1)$, using the definition of Γ.

b) Compute the Laplace transform of $\sqrt{t}$ from the definition of the transform. You can leave your answer in terms of any special functions we have studied, such as J_p or Γ.

c) Prove that if the Wronskian vanishes at some point x_0 in the domain, for two solutions f_1, f_2 of the usual Ch.4.6 DE, then the two solutions are L.D.

Bonus) [about 4 points] Find the first two nonzero terms of the McLaurin series for $(1 + 3x + 5x^2 + \cdots)^{-1}$. If you use the back, leave a note here, as usual.

Remarks and Answers: The average among the top 17 was 57, which is a bit low. The high scores were 87 and 71. The worst results were on problems 10 and 12 (21% and 36%). The best results were on problems 8 and 4 (85% and 78%). It was nice to see an improvement in series skills, though only 2 people got the bonus.

I do not create a scale specifically for the final, and have not yet combined the exam scores with the HW scores, etc.

1) $\mu = \exp(\int P) = e^{3x}$. The DE becomes $(3ye^{3x} - 3x^2)dx + e^{3x}dy = 0$. It is exact because $\frac{\partial}{\partial x} e^{3x} = 3e^{3x} = \frac{\partial}{\partial y}(3ye^{3x} - 3x^2)$. See 2.3.3.

2) $x^2y + y^3 = k$. Get $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$ (using $c = \frac{x^2+y^2}{x^3}$). Set the new $\frac{dy}{dx} = \frac{-2xy}{x^2+3y^2}$ and solve using whichever Ch.2 method you prefer (exact or homogeneous). See 3.1.7.

3) Use variation of parameters with $y = v_1 \sin(x) + v_2 \cos(x) = -\cos(x)\ \ln|\sec x + \tan x|$. Get $v_1 = -\int \frac{\tan(x)\cos(x)}{1\cdot(-1)}\, dx = -\cos(x)$, etc. See the first Example in Ch.4.4, but you can use the integral just mentioned instead of determinants (if you do not know Cramer's Rule). Note that $\tan(x)$ does not fit Ch. 4.3 (UC's) because its higher derivatives get messy.

4) 4/3 and 2/3. The DE leads to 4 series, but only three include a $k = 0$ term (assuming you use the same style and notation as in class). Setting $k = 0$ those three add to $r(r-1) - r + 8/9 = 0$, and then apply the quadratic formula (or factor it, or complete the square, etc). See 6.2.7.

5) $y(t) = \begin{cases} \sin(t) & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases}$, from $Y(s) = \frac{1+e^{-st}}{1+s^2}$. See 9.4D.3.

6) $Y = \frac{2}{s^2+4}$. See the text Example or the lecture notes for this, or a very similar calculation. You need to integrate by part twice.

Or: there was a hint to use an IVP as a shortcut (also done in class). The IVP is $y'' + 4y = 0$ with $y(0) = 0$ and $y'(0) = 2$. We know from Ch.4 that the solution is $y = \sin(2t)$. It is easy to compute Y from this IVP and get the answer above.

7) Simple algebra leads to $m = 1 \pm 2i$, repeated (there must be 4 roots). So, $y(t) = c_1 e^t \sin(2t) + c_2 t e^t \sin(2t) + c_3 e^t \cos(2t) + c_4 t e^t \cos(2t)$.

8) P.Fractions gives $A = 4$, so $y = 4e^{-2t} + \cos(t) - 3\sin(t)$

9) $\sin(3t)/6 - t\cos(3t)/2$ from the definition of $f * g$ and some integration. As a shortcut, you might use $L(f * g) = L(f)L(g)$ along with line 13 of Table 9.1 (if you can remember that). Common mistakes: confusing variables such as t and τ, or dropping the '3' at various points.

10) See Thm 9.10.

11) TFTF TFTT

12) See the text for a and c, the lecture notes for b.

The most common choice was c. Many answers were very hard to follow. I gave a few points partial credit for using words, almost any words, and for a correct Wronskian formula. I gave more for including important terms such as x_0, x, c_1 etc. Very few people got full credit for including all the key features such as the EU Thm.

B) $1 - 3x + \cdots$.