MAS 3105 Nov 27, 2017
Quiz 6 Key Prof. S. Hudson

1) [15 pts] Define an inner product on R^3 by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{3} x_i y_i w_i$ where the weight vector is $\mathbf{w} = (1/4, 1/4, 1/2)^T$. Show that $\mathbf{x} = (4, 1, -1)^T$ and $\mathbf{y} = (1, 2, 3)^T$ are orthogonal with respect to this inner product.

2) [25 pts] a) Find all Least Squares solution(s) to the system. You may use the given $A = QR$ factorization, if you like (not required).

$$\begin{aligned} x_1 &= 2 \\ x_1 + x_2 &= 4 \\ -x_1 &= 6 \\ x_1 + x_2 &= 0 \end{aligned} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

2b) Find the projection of $(2, 4, 6, 0)^T$ onto $R(A)$, maybe quickly.

3) [20 pts] Choose ONE of these to prove and CIRCLE it. Write the proof on the back.

a) Thm 5.2.1: The Fundamental Subspace Theorem (include both directions).

b) Thm 5.3.2: If A has rank n, then the normal equations have a unique solution.

c) State the definition of an *orthogonal* matrix. Show that the product of two orthogonal matrices is also orthogonal, explaining your reasoning as usual.

Remarks and Answers: The average was approx 39 / 60, with top scores of 60 and 52. Here is an advisory scale:

A's 44 to 60
B's 38 to 43
C's 32 to 37
D's 26 to 31

For an estimate of your semester grade, average your best 5 out of 6 quiz grades. If that does not match the number written on your quiz near the date then email me asap. If it matches, then use the scale below. If your HW or MHW grades are abnormal, make a small adjustment or see me for help with that.

A's 47 to 60
B's 41 to 46
C's 35 to 40
D's 29 to 34

1) $(4)(1)(1/4) + (1)(2)(1/4) + (-1)(3)(1/2) = 1 + 1/2 - 3/2 = 0$. Silly mistakes were fairly common, but you know the answer should be zero, so these should be easy to catch. A

few people tried to prove *orthonormal* but that was not required and is not true. Notice that this problem is worth only 15 points.

2a) $\hat{x} = (-2, 4)^T$. Since A has rank 2, there is only one solution. You can solve the usual $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ (ignoring the hint) or use the slightly faster method, $R\hat{\mathbf{x}} = Q^T\mathbf{b}$.

2b) $\mathbf{p} = A\hat{x} = (-2, 2, 2, 2)^T$. If you didn't get 2a, you could also use the slightly longer method $\text{proj}_S\ \mathbf{b} = (\mathbf{b}^T\mathbf{q_1})\mathbf{q_1} + (\mathbf{b}^T\mathbf{q_2})\mathbf{q_2}$. Roughly, 2a was worth 15 points and 2b was worth 10.

3) For 3a and 3b, see the text or lectures. For 3c the main idea is $(AB)^T AB = B^T A^T AB = B^T B = I$ using a key property of orthogonal matrices. You need to state the definition too, but you don't need that in the proof.

For 3b, the main idea is to show $A^T A$ is nonsingular by studying its rank, as in HW 5.2.13 (or the textbook proof). For full credit, I expected you to go through most of 13abcd, maybe omitting some minor steps.