MAS 3105 | Nov 14, 2019
Quiz 6 | Prof. S. Hudson

1) [40 points] Let S be the subspace of R^3 spanned by $\mathbf{v} = [1, 1, -1]^T$.

1a) Find a basis for $S^\perp$.

1b) Write the vector $\mathbf{w} = [2, 1, 0]^T$ as a sum of two vectors, one from S and one from $S^\perp$.

2) [30 points] Find the least squares solution to the system.

$-x_1 + x_2 = 10$
$2x_1 + x_2 = 5$
$1x_1 - 2x_2 = 20$

3) [30 points] Choose ONE. You may continue on the back, but leave a note here.

a) State and then prove the normal equations used with Least Squares problems.

b) $N(A^T A) = N(A)$ (this includes parts a and b of exercise 5.2.13).

c) State the definition of a *norm* and show that $||\mathbf{x}||_1 = |x_1| + |x_2|$ (Manhattan distance) is a norm on R^2.

Remarks, Scales: The quiz average was approx 57, with top scores of 90 and 80 out of 100. The advisory scale for Quiz 6:

A's 66 to 100
B's 56 to 65
C's 46 to 55
D's 36 to 45

Your current average is written in the upper right on your quiz. This includes your best 5 out of 6 quiz grades, based on the assumption that your MHW average will replace your lowest quiz grade. The class average for that number is about 73. As before, I have not yet used the HW or MHW in the setting this advisory semester scale:

A's 81 to 100
B's 71 to 80
C's 61 to 70
D's 51 to 60

Answers:

1a) The standard method (for harder versions of this problem) is to choose a matrix A so that $S = R(A)$ and use the theorem that $S^\perp = N(A^T)$, and then GE. But I accepted trial-and-error methods, usually leading to a basis such as $\{[1,\ -1,\ 0]^T, [1,\ 0,\ 1]^T\}$. Other answers are possible.

We have a theorem that dim $S^{\perp}$+ dim $S = n$, so you should expect two vectors in the basis. I did not give much credit for finding just one.

1b) $\mathbf{w} = \mathbf{v} + [1,\ 0,\ 1]^T$. Again, I accepted trial and error methods, which were pretty easy if you found the basis given above for (1a). There is only one correct answer to (1b), based on the $S \oplus S^{\perp}$ theorem.

For harder versions (or if you found some unusual answer to (1a)), the standard method is to set $\mathbf{w} = x_1\mathbf{v} + x_2\mathbf{b_1} + x_3\mathbf{b_2}$ and solve a system to find $\mathbf{x}$. See the proof of thm 5.3.2, if needed.

2) The normal equations, $A^TA\mathbf{x} = A^T\mathbf{b}$ are

$$\begin{pmatrix} 6 & -1 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 20 \\ -25 \end{pmatrix}, \text{ so } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{35} \begin{pmatrix} 95 \\ 130 \end{pmatrix}.$$

The last step is probably easiest using Cramer's Rule or $(A^TA)^{-1}$, since the system is so small, but GE is also OK.

3a) You were supposed to prove that $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. See pages 227-228.

3b) You can prove (i) that $N(A) \subseteq N(A^TA)$ which is very easy, then prove (ii) that $N(A^TA) \subseteq N(A)$ which is slightly harder - see exercise 5.2.13 a+b, or the first 3 sentences in the proof of Thm 5.3.2.

3c) See the text for the definition of norm (the 3 rules), and recall that $||\mathbf{x}||_1 = |x_1| + |x_2|$. To prove rule 2 for example, write $||\alpha\mathbf{x}||_1 = |\alpha x_1| + |\alpha x_2| = |\alpha||x_1| + |\alpha||x_2| = |\alpha|(|x_1| + |x_2|) = |\alpha|\ ||\mathbf{x}||_1$ perhaps with some short explanations.