1a) [each part of problem 1 is 5 points]. In the Rabbit example, we studied $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L([a, b]^T) = [a+b, 2a]^T$. Find the matrix representation $A$ of $L$ (w.r.t. the std basis of $\mathbb{R}^2$).

1b) We found two eigenvectors of $A$, $x_1 = [1, 1]^T$ and $x_2 = [1, -2]^T$, which form a basis $X$ of $\mathbb{R}^2$. Find the corresponding eigenvalues.

1c) Find the matrix representation $B$ of $L$ w.r.t. the basis $X$.

1d) Find an eigenvector for $B$. Is it also an eigenvector of $A$? Explain briefly.

1e) Let $x = [2, 3]^T_X$ (the coordinates are wrt the eigenvector basis, $X$). Find $[L(x)]_S$, in standard coordinates.

2) [20 pts] True-False. You can assume the matrices are all square.

- $A^T A$ and $AA^T$ always have the same rank.
- If $A$ and $B$ are unitary then $AB$ is unitary.
- If $A$ is singular then $AB$ is singular.
- If $A^H = -A$ then $A$ is normal.
- If $A$ is unitary then $A$ is not defective.
- If $A$ is Hermitian and unitary then $A^2 = I$.
- If $U$, $V$ are subspaces of $\mathbb{R}^n$, and $U \perp V$ then $V \subset U^\perp$.
- If $A$ is unitary and $x \in C^n$ then $||Ax|| = ||x||$.
- If $\lambda$ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.
- If $\lambda$ is an eigenvalue of a unitary matrix then $\lambda$ is real.

3) [10pts] Suppose $A$ is a 4x4 matrix and det $A = 3$. Find det (adj($A$)).

4) [10pts] Choose ONE of these to prove.

- a) If an nxn matrix $A$ is diagonalizable, then it has n L.I. eigenvectors.
- b) State and prove the Spectral Theorem.

5) [15pts] You are given this $A = QR$ factorization to help with the questions below. Note that the $\frac{1}{2}$ scalar is part of $Q$. 

1
\[
\begin{bmatrix}
1 & -1 & 4 \\
1 & 4 & -2 \\
1 & 4 & 2 \\
1 & -1 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
1 & -1 \\
1 & 1 \\
1 & 1 \\
1 & -1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
2 & 3 & 2 \\
0 & 5 & -2 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

5a) Find a Least Squares solution to the system \( Ax = b = [1, 2, 3, 4]^T \)

5b) Find \( p = \text{proj}_{R(A)} b \).

5c) Let \( S = \text{span} \{ q_1, q_2 \} \subset R(Q) \subset R^4 \). Find \( \text{proj}_S b \).

6a) [10pts] Diagonalize \( A \). The eigenvalues are \( \pm 1 \):

\[
A = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

6b) [5pts] Find a matrix \( B \) so that \( B^2 = A \). This may involve your answer to 4a) and may involve complex numbers. If your work results in a product or sum of matrices, you do not have to simplify (but this might help you check your answer).

6c) [5pts] Find \( e^A \) (ditto remarks in 4b).

Remarks and Answers: Nobody handed this in early, so I guess it was a bit too long for an 80 minute time period. On the other hand, many people wasted time on calculations they didn’t really need (eg, in problem 1, or finding \( R^{-1} \) in problem 5).

The average grade was about 55 / 100. The worst results were on problem 5 (especially part c) and 3. You can adjust the usual scale about 10 points downwards for this exam. I have not yet calculated the scale for the semester grades.

1) The first 3-4 answers are in your lecture notes and do not require any real work.

   a) \( A = \begin{bmatrix} 1 & 1; 2 & 0 \end{bmatrix} \) (in MATLAB notation).

   b) 2 and -1 (if you forgot these numbers, just multiply \( Ax_1 \) to find \( \lambda_1 \), etc.)

   c) \( B = D = \begin{bmatrix} 2 & 0; 0 & -1 \end{bmatrix} \).

   d) Since \( B \) is diagonal, it has eigenvectors \( [1, 0]^T \) and \( [0, 1]^T \), which are not the same as the eigenvectors of \( A \), at least on the surface. But the coordinates in these vectors are wrt the basis \( X \), so the first one is actually \( x_1 \), so it is an eigenvector of \( A \) in disguise (same idea for the second one). Since this was rather tricky, I gave as much partial credit as possible.

   e) \( [1, 10]^T \). You can multiply by \( B = D \) to do \( L \), and then by \( X \) (the transition matrix you get from the basis \( X \)) to convert to the std basis. Or, you can multiply by \( X \) first and then by \( A \). These give the same result since \( AX = XD \) (see Ch 6.3).
2) TTTTT TTFTF

3) $3^{4-1} = 27$. If you forgot the 4, recall the HW about $\det(\alpha A)$. Also, since this didn’t ask for a proof, you could work out an example. You could choose $A$ to be a diagonal matrix with nonzero entries 1,1,1,3. Then $\text{adj} \ A$ is pretty easy to compute (diag with entries 3,3,3,1).

4) See text. Most people chose b) and did it pretty well. Some answers to a) were not well-organized; you should start from $A = XDX^{-1}$, and explain why $X$ contains n LI evecs.

5a) $x = [2.9,-0.1,-0.25]^T$. Start from $Rx = QTb$, a simple system that’s ready for back substitution. This is one of the reasons for doing a $QR$ factorization. Several people converted this to $x = R^{-1}QTb$, which takes much much longer.

5b) Use the answer to 5a), $p = Ax = [3, 2, 3, 2]^T$.

5c) $[5,5,5,5]^T$. This is not very related to the previous parts, but it does use the fact that the two columns of $Q$ must be onl. Based on the theorem in Ch 5.5, add the projections onto $q_1$ and $q_2$. By coincidence, $b \perp q_2$, so you could just project onto $q_1$ (but explain why).

Remark: It’s possible to do 5c) like 5b), or to do 5b) like 5c), but I think the explanations above are the simplest.

6a)

$$ A = XDX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} $$

This is a routine Ch 6.3 problem (though many people missed it). The given eigenvalues go into $D$, the eigenvectors into $X$. For example, $x_1 = [1, 1]^T$ comes from $N(A - I)$. Other answers are possible.

6b) $B = A^{1/2} = XD^{1/2}X^{-1}$ where

$$ D^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} $$

The notation $D^{1/2}$ conveys the right idea, but it is not technically correct, since there are many matrices that work as well as the one I gave (eg, you can replace the 1 by -1, or the $i$ by $-i$). There was a HW like this, but expressed more accurately (see text).

6c) $e^A = Xe^D X^{-1}$ where

$$ e^D = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix} $$

which you did not have to simplify further.