MAS 3105 April 28, 2011
Final Exam and Key Prof. S. Hudson

1) Suppose L rotates each vector in R^2 by 45 degrees clockwise. Find the matrix representation of L (standard basis). Hint: $\cos(45) = \sqrt{1/2}$.

2) [20 pts] True-False. You can assume the matrices are all square.

All the eigenvectors of a nilpotent matrix are 0.

If eigenvectors $\mathbf{x_1}$ and $\mathbf{x_2}$ correspond to $\lambda_1 \neq \lambda_2$ then $\mathbf{x_1} \perp \mathbf{x_2}$.

If A is similar to B then they have the same rank.

The Google PageRank algorithm is a Markov process.

$\exists A \in R^{3\times 3}$ such that $(3,1,1) \in \text{Row}(A)$ and $(1,1,3)^T \in N(A)$.

If $L : V \to W$ is linear, then $\ker(L) \perp L(V)$.

We used a basis of 3 eigenvectors to solve the Rabbit problem.

For all 3×3 matrices, rank $(AB) \leq$ rank (B).

The normal equations are $AA^T\mathbf{x} = A^T\mathbf{b}$.

Overdetermined systems are usually inconsistent.

3) Choose ONE [from Ch 6.4 HW]

a) Show that if U is unitary and $\mathbf{x} \in C^n$, then $||U\mathbf{x}|| = ||\mathbf{x}||$.

b) Show that if $\mathbf{x} \in C^n$ is a unit vector and $U = I - 2\mathbf{x}\mathbf{x}^H$, then U is unitary and Hermitian, and $U^{-1} = U$.

4) Choose ONE of these to prove.

a) If an nxn matrix A is diagonalizable, then it has n L.I. eigenvectors.

b) State and prove the Spectral Theorem.

5) [15pts] You are given this $A = QR$ factorization to help with the questions below. Note that the $\frac{1}{2}$ scalar is part of Q.

$$\begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

5a) Find a Least Squares solution to the system $A\mathbf{x} = \mathbf{b} = [1,2,3,4]^T$

5b) Find $\mathbf{p} = \text{proj}_{R(A)}\mathbf{b}$.

5c) Let $S = \text{span}\ \{\mathbf{q_1}, \mathbf{q_2}\} \subset R(Q) \subset R^4$. Find $\text{proj}_S \mathbf{b}$.

6) [5 pts each]

a) Define linearly independent.

b) State the parallelogram law about norms of vectors (from MHW).

7) Compute e^A (and simplify) given that $A = XDX^{-1}$ and

$$X = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

8) [15 pts total] Give an example of each:

a) A normal matrix that is not Hermitian.

b) A defective matrix.

c) A matrix A with no zero entries such that $A^2 = I$. Hint: you might start from $A = XDX^{-1}$, choose D carefully, and guess at X.

Remarks and Answers: The average was about 55 / 100, which is low. The problems on recent topics had low average results, but problems 1, 2 and 4 got good results. Most the problems were new, but Problem 5 has appeared on other finals.

1) Use $L(e_j)$, as usual;

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

2) TFTTF FFTFT

3) Most people chose a), which is easy, but the results weren't very good. Main steps of a):

$$||Ux||^2 = (Ux)^H Ux = x^H U^H Ux = x^H x = ||x||^2$$

4) See text.

5a) $x = [2.9, -0.1, -0.25]^T$. Start from $Rx = Q^T b$ which is ready for back substitution. This is one of the reasons for doing a QR factorization. Several people converted this to $x = R^{-1}Q^T b$, which takes much much longer.

5b) Use the answer to 5a), $p = Ax = [3, 2, 3, 2]^T$.

5c) $[5/2, 5/2, 5/2, 5/2]^T$. This is not much related to the previous parts, but it does use the fact that the two columns of Q must be onl. Based on the theorem in Ch 5.5, add the projections onto q_1 and q_2. By coincidence, $b \perp q_2$, so you could just project onto q_1 (but explain why).

Remark: It's possible to do 5c) like 5b), or to do 5b) like 5c), but I think the explanations above are the simplest. This problem (or similar) has appeared on some of my past exams.

6)a) See text. b) See MHW (nobody got this one).

7)

$$e^A = Xe^D X^{-1} = \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$$

8a) [Normal means $A^H A = AA^H$]. You can choose any unitary matrix or any skew-Hermitian one (etc), such as

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

If it is not fairly obvious that your matrix is normal, you should justify that by a calculation or an explanation.

8b) Defective means not diagonalizable. From class:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

8c) To my surprise, several people found a good answer without any clear strategy. One person chose A to represent a reflection of R^2 in a line (they used $\theta = -\pi/8$ but almost any such line is OK). Not sure they did this on purpose, though. My hint was to set $A = XDX^{-1}$, and then

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $A^2 = D^2 = I$ regardless of X. Then I chose X randomly, hoping that A would not contain zeroes. I used

$$X^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and got} \quad A = \begin{pmatrix} 5 & 12 \\ -2 & -5 \end{pmatrix}$$

but there are many other answers.