MAS 3105 April 25, 2013

Final Exam and Key Prof. S. Hudson

1) [10pts] Suppose A is a 6×7 matrix and its column space is spanned by $\mathbf{a_1}$ and $\mathbf{a_3}$, which are linearly independent (LI). Find the dimension of $N(A)$ and explain briefly.

2) [15 points] Choose ONE of these to prove (you can use the back).

a) If an nxn matrix A is diagonalizable, then it has n L.I. eigenvectors.

b) Prove that $N(A) = R(A^T)^{\perp}$.

c) Prove that det AB = det A det B. To save time, you can assume they are both nonsingular.

3) [15 pts] Let

$$A = \frac{1}{3}\begin{pmatrix} 1 & 8 \\ 2 & 7 \\ 2 & -2 \end{pmatrix} \qquad and \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

a) Use the Gram-Scmidt process to find an orthonormal basis for $R(A)$.

b) Factor A into QR (with the usual meaning).

c) Use b) to solve the least squares problem $Ax = b$ quickly.

Show all your work, but you can check some of your steps against some of these: $||a_1|| = 1$, $||a_2|| = 3.6$, $< a_1, a_2 >= 2$, $a_2 - 2a_1 = [2\ 1\ -2]^T$, and $a_1 - 2a_2 = [-5\ -4\ 2]^T$.

4) [20 pts] True-False. You can assume the matrices are square unless stated otherwise.

A and A^2 always have the same rank.

A and A^T always have the same eigenvalues.

If A is nonsingular then AB is nonsingular.

If A is normal then A is unitary or Hermitian.

If A is unitary then A^2 is unitary.

If A represents a rotation then A is an orthogonal matrix.

Every 3x3 matrix has at least one real eigenvalue.

If A is a 7x5 matrix, then $Ax = 0$ has nontrivial solutions.

If A is singular, then its nullity is positive.

If 0 is an eigenvalue of A, then A is singular.

5) [15 pts: HW 6.3.23] For maximal credit (and less work) you are to find the eigenvalues of the matrix A below without computing $p(\lambda)$. For less credit, you can use $p(\lambda)$ instead.

a) It is a transition matrix for a Markov process. Predict one eigenvalue from that.

b) Compare the rows of A to determine a second eigenvalue. Explain.

c) Compute the trace of A. Use that to determine a third eigenvalue.

$$A = \begin{pmatrix} \frac{2}{4} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \end{pmatrix}$$

6) [10 pts] Let

$$\mathbf{z_1} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \quad \text{and} \quad \mathbf{z_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{z} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix}$$

a) Show $\mathbf{z_1}$ and $\mathbf{z_2}$ form an orthonormal set in C^2.

b) Write the vector $\mathbf{z}$ as a linear combination of $\mathbf{z_1}$ and $\mathbf{z_2}$.

7) [10 pts] a) Find the eigenvalues of A. Factor $A = XDX^{-1}$.

b) Can A also be diagonalized by a unitary matrix ? Either do it, or explain why this is impossible.

$$A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 15 & 4 \end{pmatrix}$$

8) [5pt] Find a basis of $N(A)$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 2 & 1 & 3 & 2 \end{pmatrix}$$

Remarks and Answers: The average was about 63, including almost everybody now, with high scores of 93 and 91. The average scores were pretty normal on all problems, except maybe for the 46% on 6) and 80% on 8). I am still collecting late HW scores, etc, and have not yet set a scale for the semester.

1) Based on the rank-nullity theorem, 7 - 2 = 5.

2) See the text. Most people chose (c).

3) a) $\mathbf{q}_1 = \mathbf{a}_1$ and $\mathbf{q}_2 = \frac{1}{3}[2, 1, -2]^T$.

b) Based on the work in (a), $r_{11} = 1$, $r_{12} = 2$, $r_{22} = 3$.
c) $\hat{x} = [1, 0]^T$.

4) FTFFT TTFTT

5) 1, 0, 7/30. The trace is 37/30 (only 3 people added the fractions correctly and got this right).

6) a) Check both norms and $\perp$. For example, check that $\mathbf{z_1}^H \mathbf{z_2} = 0$.

b) $\mathbf{z} = 4\mathbf{z_1} + 2\sqrt{2}\mathbf{z_2}$. Show some work, such as $\mathbf{z_1}^H \mathbf{z} = 4$. It is also OK to set-up and solve a linear system, but since the basis is orthonormal, the inner product method (from Ch.5) is OK and faster.

7) a) $p(\lambda) = (7 - \lambda)(\lambda^2 - 6\lambda - 7)$, so $\lambda = 7, 7, -1$. The $\mathbf{x}_i$ for 7 are $[1, 0, 0]^T$ and $[0, 1, 5]^T$, and for -1 it is $[0, 1, -3]^T$ (as usual, other answers are possible for these). The rest, about XDX^{-1}, is now easy. In hindsight, I didn't explicitly ask you to compute X^{-1}. If you did so correctly, I gave 2 extra points.

It is not diagonalizable by U because it is not *normal* (you can easily check that $A^H A \neq AA^H$, see Ch.6.4). I also gave credit for saying that the eigenvectors you found are not orthogonal, though this is not completely convincing (what if someone else found different ones which ARE orthogonal ? With the repeated eigenvalue, this idea requires more explanation.).

8) $[1, 0, 0, -1]^T$ and $[-3, 3, 1, 0]^T$ (other answers are possible).