MAS 3105 April 23, 2014
Final Exam and Key Prof. S. Hudson

1) Let $S = \text{span }\{(1,2,3,0)^T, (1,0,0,1)^T\}$, a subspace of R^4. Find a basis of $S^\perp$. For about 5 points extra credit, find an ONL basis of it.

2) [15 pts] Suppose a Markov process has the transition matrix A below. a) Find its characteristic polynomial and eigenvalues the usual way (without using part b below).

$$A = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$$

b) Suppose the process starts with $\mathbf{x_0} = [1,0]^T$. Find the steady state probability vector $\mathbf{x}$. You can use the factorization below to save some time (you can get λ_2 from part a). If you use a shortcut (eg, you do not compute a limit), explain it.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix}$$

3) [15pts] a) Find the matrix rep A of $L : R^2 \to R^2$, where L is a 45 degree CCW rotation.

3b) Find A^{100}. Either show your work or explain your reasoning.

4) You are given three data points for (x, y): $(0,0), (1,7)$ and $(2,2)$. You may put these into a chart if you like. Find the best least squares fit by a linear function, $y = c_0 + c_1x$.

5) [20 points] Answer True or False:

Every 2x2 matrix has at least one real eigenvalue.

If A and B are similar nxn matrices, then det A = det B.

Every square matrix A can be factored, $A = QR$, as an orthogonal matrix times a nonsingular upper triangular matrix.

If A is a 7x5 matrix, then $Ax = 0$ has nontrivial solutions.

If A is nxn and singular, then its nullity is positive.

If $L : R^2 \to R^2$ is linear, and $\mathbf{x} \perp \mathbf{y}$ in R^2, then $L(\mathbf{x}) \perp L(\mathbf{y})$.

If 0 is an eigenvalue of A, then A is singular.

Every real orthogonal matrix is unitary.

Every diagonal matrix is normal.

Every Hermitian matrix is diagonalizable.

6) Let V be the space of 2×2 symmetric real matrices. Find a basis of V and the dimension of V.

7a) Find a basis for $N(A-2I)$, where $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

7b) Use 7a to decide whether A is diagonalizable, and justify your answer (this should not require much more calculation).

8) Choose ONE: [or, for less credit, you can prove the det AB theorem instead]:

A) State and prove the Spectral Theorem (6.4.4).

B) State and prove Thm 5.6.3 about solving Least Sqs problems, given $A = QR$.

C) Use induction to prove that if a square A has 2 identical rows then det $A = 0$.

Bonus [5 pts]: Give an example of an inner product on the space P_3 that does not involve an integral.

Remarks and Answers: The high scores were 110, 107 and 101, very nice ! The average was about 70 out of 100, not counting 3 scores below 20. The results were mostly good on all the problems except for problem 6 (43%). I will not set a scale for the final. I have not done that yet for the semester.

1) Compute $N(A)$ using GE, where the first row of A is (1,2,3,0), etc. I get $N(A) =$ span $\{(-1, 0.5, 0, 1)^T, (0, -1.5, 1, 0)^T\}$, but there are other answers. For the EC, use GSO.

2) $p(\lambda) = \lambda^2 - 3\lambda/4 - 1/4 = 0$, $\lambda_1 = 1$, $\lambda_2 = -1/4$.

2b) The fastest answer is to take the first eigenvector, the first column in the SDS^{-1} above, but "normalize" it to be $[2/5, 3/5]^T$. With slightly more work, you can get the same thing from $\lim_{n\to\infty} A^n \mathbf{x_0}$.

3a) is below. 3b) is -I (because $A^8 = I$ and $A^4 = -I$).

$$A = \sqrt{1/2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

4) $y = 2 + x$. See Ch.5.3 (and a similar question from the previous exam.)

5) FTFFT FTTTT

6) See Ch.3 and HW. Many people tried to answer with column vectors, but V contains matrices and the basis should be a list of matrices, too.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

7a) $[1, 0, 1]^T, [0, 1, 0]^T$

7b) Yes, we get 3 LI eigenvectors.

8) Most people chose A and did pretty well.

Bonus [5 pts]: Two people used $\langle ax^2 + bx + c, dx^2 + ex + f \rangle = ad + be + cf$.