MAS 3105 Dec 7, 2016

Final Exam Prof. S. Hudson

1) [15pts] Let $\mathbf{b_1} = [1, 1, 0]^T$, $\mathbf{b_2} = [1, 0, 1]^T$, and $\mathbf{b_3} = [0, 1, 1]^T$. Let L be the linear transformation from R^2 to R^3 defined by

$$L(\mathbf{x}) = 3x_1\mathbf{b_1} + x_2\mathbf{b_2} + (x_1 - x_2)\mathbf{b_3}$$

1a) Find the matrix A representing L with respect to the bases $\{\mathbf{e_1}, \mathbf{e_2}\}$ and $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}$.

1b) Find a basis for the range of L. Write the vector(s) in standard coordinates.

2) [10 pts] Use the Gram-Scmidt process to find an orthonormal basis for $R(A)$, where

$$A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \\ 2 & -2 \end{pmatrix}$$

3) [10 pts] Choose ONE. Both are slightly related to your MHW, but you can probably solve either one without having done that.

a) State and prove the parallelogram law (using linear algebra methods, not geometry). It is a formula a bit like this incorrect version:

$$||\mathbf{x}||^2 + ||\mathbf{y}||^2 + ||\mathbf{x} + \mathbf{y}||^2 = 3||\mathbf{x} - \mathbf{y}||^2$$

b) Let A be a 7×7 matrix, with every entry equal to 1, except that each $a_{jj} = 2$ (so, it has 2's on the diagonal). List the eigenvalues of A by multiplicity. For full credit, first find the rank of $A - I$, and use that info to get an answer quickly, without using $p(\lambda)$.

4) [20 pts] True-False. You can assume the matrices are square unless stated otherwise.

- If A represents a rotation then A is an orthogonal matrix.
- If A is row equivalent to a diagonalizable matrix, then A is also diagonalizable.
- If A is a 5x6 matrix, then $Ax = 0$ has nontrivial solutions.
- A and A^2 always have the same rank.
- If A is upper triangular then it is diagonalizable.
- If A is singular then AB is singular.
- If A is unitary or Hermitian then A is normal.
- If A is unitary then A^3 is unitary.
- If A is singular, then it does not have full rank.
- If Q is orthogonal, then $\det Q = 1$.

5) [10 pts] Find a nice 3x3 coding matrix A as described in Ch. 2.3. For full credit, it should:

have only integer entries, mostly nonzero, and
have an inverse with only integer entries.

If you are sure of your method, you do not have to compute A^{-1} explicitly, and you can write A as a sum or product of matrices without simplifying (but include comments).

6) [15 pts] For maximal credit (and less work) you are to find the eigenvalues of the matrix A below without computing $p(\lambda)$. For less credit, you can use $p(\lambda)$ instead.

a) Given: it is a transition matrix for a Markov process that converges to a steady state vector. Predict one eigenvalue from that.

b) Compare the rows of A to determine a second eigenvalue. Explain.

c) Compute the trace of A. Use that to determine a third eigenvalue.

$$A = \begin{pmatrix} \frac{2}{4} & \frac{1}{3} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{2}{5} \end{pmatrix}$$

7) [10 points] Choose ONE of these to prove.

a) If an nxn matrix A is diagonalizable, then it has n L.I. eigenvectors.

b) Prove that if A is similar to B then they have the same rank.

8) [10 pts] Let

$$\mathbf{z_1} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \quad \text{and} \quad \mathbf{z_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{z} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix}$$

a) Show that $\mathbf{z_1}$ and $\mathbf{z_2}$ form an orthonormal set in C^2.

b) Write the vector $\mathbf{z}$ as a linear combination of $\mathbf{z_1}$ and $\mathbf{z_2}$. Suggestion: use a Ch.5 theorem to find the 2 coefficients quickly (no GE needed).

Bonus: Give an example of a matrix A such that $A^4 = O$ but $A^3 \neq O$.

Remarks and Answers: The average was 62 / 100, with high scores of 105 and 90. The lowest average scores were on problems 1, 3, and 7 (approx 45%). The highest were on 2 and 5 (approx 85%). I have not set the semester scale, but it will probably be a bit lower than the one on the syllabus.

1a) This is HW 4.2.6 and does not use the given coordinates of the $\mathbf{b_j}$, only the formula for L.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

1b) $\{[3,4,1]^T, [1,-1,0]^T\}$. This was the most common answer, but any 2 LI vectors in $R(A)$, changed to standard coordinates, will work.

2) $\{\frac{1}{\sqrt{5}}[1,0,2]^T, \frac{1}{\sqrt{21}}[4,1,2]^T\}$ This one is a bit easy because $\mathbf{a_1} \perp \mathbf{a_2}$, so you really just have to normalize them. But if you didn't notice that, your GSO calculations should lead to $\mathbf{p} = \mathbf{0}$, with the same final result.

3a) The correct version is $||\mathbf{x}+\mathbf{y}||^2 + ||\mathbf{x}-\mathbf{y}||^2 = 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2$. Your proof might start with $||\mathbf{x}+\mathbf{y}||^2 = (\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) = \cdots$, etc.

3b) 1,1,1,1,1,1,8. Since $A - I$ (imagine that $\lambda = 1$) is full of 1's, it's column space is spanned by one vector, so rank $(A - I) = 1$. So, nullity $(A - I) = 7 - 1 = 6$ and we can get 6 LI eigenvectors for $\lambda = 1$. We have a theorem that trace $A = \sum \lambda_j$, so $14 = 6 + \lambda_7$, so $\lambda_7 = 8$.

4) TFTFF TTTTTF

5) The main thing is to make $\det(A) = 1$ so that $A^{-1} = \text{adj}A$ which will have only integer entries. The standard method to make $\det(A) = 1$ is to multiply several type III matrices. For example,

$$A = E_1E_2E_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

(and stop). Many people found an acceptable matrix A through trial and error, and got full credit. However, this is not very efficient, and there were several calculation errors trying to make det $A = 1$.

6a) 1. 6b) 0, because 2 rows are identical, so that A is singular. 6c) trace = 37/30, so $\lambda_3 = 7/30$.

7) See the text for 7a. For 7b, recall that with nonsingular matrices S and T, rank of SBT = the rank of B (see Ch. 3). Apply this with $T = S^{-1}$, so that $A = SBS^{-1}$ has the same rank as B.

8a) Check both norms equal 1 and that the inner product $\mathbf{z_2}^H\mathbf{z_1} = 0$.

8b) $\mathbf{z} = 4\mathbf{z_1} + 2\sqrt{2}\mathbf{z_2}$

Bonus) Schur's theorem (etc) suggests looking for an upper triangular example with zeroes on the diagonal. I think A must be at least 4x4 (proof?). Anyway, here is one example. It is worthwhile to compute A^2, A^3 and A^4 to see how this works.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$