MAS 3105 | Dec 13, 2017
Final Exam | Prof. S. Hudson

The first five problems are 7 points each and the last ones are 10 points each.

1) Use Cramer's Rule to find x_1, given that
$x_1 + x_2 = 5$
$3x_1 - 2x_2 = 0$

2) Solve for the 2x2 matrix X given that $UX + I = B$, U is unitary, and

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & 1+i \\ 1-i & -2 \end{pmatrix} \qquad B = \begin{pmatrix} i & 2 \\ 3+i & 0 \end{pmatrix}$$

Typo corrected 12/15/17: The $\sqrt{6}$ was originally a 2. As far as I can tell, this did not affect anyone's method or their grade.

3) Let V be the space of 2×2 symmetric real matrices. Find a basis of V and the dimension of V.

4) Suppose the eigenvalues of a 3x3 matrix A are 5, 2 and -1. What are the eigenvalues of $B = (A - I)^2$?

5) Prove that if A is nonsingular then $A^T A$ is nonsingular too. State clearly any theorems, definitions or homework results that you use.

6) A matrix A is *idempotent* if $A^2 = A$.
a) Show that every eigenvalue of an idempotent matrix is either 0 or 1.

b) Give an example of a 2x2 idempotent matrix that is not a diagonal matrix. Hint: a matrix P that represents a projection may be idempotent.

7) [15 points] Answer True or False. In each, suppose $A \in R^{3\times 3}$ is nonsingular (but not necessarily the same matrix in each part).

If A is defective then A^{-1} is defective.

$A + I$ is singular if and only if $A^{-1} + I$ is singular.

A^{-1} is similar to some upper triangular matrix T.

A is similar to some orthogonal matrix Q.

$A^T A$ is similar to some nonsingular diagonal matrix D.

8) Suppose a Markov process has the transition matrix A below, with a diagonalization provided to save you some time.

$$A = \begin{pmatrix} 1/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1/4 \end{pmatrix} \begin{pmatrix} 1/5 & 1/5 \\ 3/5 & -2/5 \end{pmatrix}$$

Suppose the process starts with $\mathbf{x_0} = [1, 0]^T$. Find the resulting steady state probability vector $\mathbf{x}$ and explain your reasoning.

9a) Find the matrix representation A of $L : R^2 \to R^2$, where L is a 120 degree CCW rotation. Hint: $\cos(120) = -1/2$.

9b) Find A^{100}. Either show your work or explain your reasoning.

10) Let $S = \text{span } \{(1, 2, 3, 4)^T, (1, 0, 0, 4)^T\}$, a subspace of R^4. Find a basis of $S^\perp$.

11) Choose ONE: as usual, include enough *words*.

A) State and prove the Spectral Theorem (Thm 6.4.4).

B) Prove part of Thm 6.3.2 - if A has n LI eigenvectors, then it is diagonalizable.

C) Use induction to prove that if a square A has 2 identical rows then $\det A = 0$.

Remarks and Answers: The average was approx 60 out of 100 with top scores of 95 and 91. The scores were slightly lower on problems 3, 6 and 8. I do not set a separate scale for the final, but include it in the overall scale, to be set asap.

1) $x_1 = \frac{-10}{-5} = 2$. Advice: check your answer when easy. For example, get $x_2 = 3$ the same way and put (2,3) into the system. I do not require this, but am a little tougher on partial credit when checking an answer is so easy to do.

2) Since $U^{-1} = U^H = U$, we get $X = U(B - I) = \frac{1}{\sqrt{6}} \begin{pmatrix} 6i & 3 - i \\ -6 & 4 - 2i \end{pmatrix}$.
Remarks: this is similar to some Ch.1 homework exercises, but maybe easier since $U^{-1} = U$. I gave about 5 points out of 7 for getting to $X = U(B - I)$. After that, most people had real trouble with the complex numbers.

A few people tried to compute the x_{ij} from a linear system. While not exactly wrong, you rarely want to go from an elegant setting to a tedious technical one. Especially true for larger problems or ones with complex numbers. Likewise, you really don't want to compute U^{-1} here. The remark that U is unitary was a hint that you don't have to.

This problem had a minor typo (see above). Full credit for 1/2 instead of $\frac{1}{\sqrt{6}}$ in the final answer.

By coincidence, this U is Hermitian as well as unitary, but this has almost no effect on the method.

3) $V = \{\begin{pmatrix} a & b \\ b & c \end{pmatrix}\} = \{a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$, which gives a basis (starting with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, etc). Dim =3.

There was a bit of confusion about the basics, that $V \subset R^{2\times2}$ is a vector space like P_3, etc, *not* containing *column* vectors. But if you like column vectors, you can use this basis to get a coordinate system for V and then write the matrices as column vectors.

4) This combines two fairly easy exercises based on the definition of eigenvalue. The eigenvalues of $A - I$ are 4, 1 and -2. The eigenvalues of $(A - I)^2$ are 16, 1 and 4.

One student found a clever "cheat" that I accepted. Let A be a diagonal matrix with diagonal entries $a_{ii} = 5$, 2, -1. Then, just work out $(A-I)^2$. Easy. This student is trusting me, I guess, that the answer doesn't depend on how he chooses A. This is not a very good general method for doing math, but it is good to think outside the box sometimes.

5) There were many different approaches to this (rank, TFAE, etc), mostly OK, but often with gaps in the logic. Here is one of the simplest (explanation left to you).

Proof: $\det A^T A = \det A^T \det A = \det A \det A \neq 0$.

6a) I thought 6a would be easy and 6b harder, but most people had more trouble with 6a. I am not aware of any theorem (about diagonal matrices for example) that makes the proof easy, but the definition of eigenvalue works well.

Proof: Let λ be an eigenvalue of A. So, $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Multiply by A, and get $A^2\mathbf{x} = \lambda A\mathbf{x}$, so $A\mathbf{x} = \lambda^2\mathbf{x}$. With the first equation, this implies $\lambda\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x} \neq \mathbf{0}$, we get $\lambda = \lambda^2$. Basic algebra shows $\lambda = 0$ or 1.

6b) Many people seemed to find simple examples by guesswork, such as $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

If you prefer a more systematic approach, you can follow the hint. Projection onto the x_1 -axis leads to a diagonal matrix, unfortunately. But projection onto the line $x_1 = x_2$ gives another good example, $A = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It is also reasonable to start from $A = XDX^{-1}$ but this may take more work.

7) TTTFT. Very brief explanations / hints: think about diagonalizable instead of defective, $A(I + A^{-1}) = I + A$, Schur's Thm, $\det Q = \pm 1$, Hermitian.

8) $[2/5, 3/5]$. The fastest method is to "normalize" x_1 (which works here because $\lambda_1 = 1$ does not repeat). Some people used $A^\infty \mathbf{x_0}$, which is bad notation, but an OK idea.

9a) $A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$.

9b) Since $360/120 = 3$, $A^3 = I$ and $A^{99} = I$ and $A^{100} = A$. In theory, a diagonalization should also do it, but expect complex numbers and much more work.

10) $[4, 0, 0, -1]^T$ and $[0, 3, -2, 0]^T$. Use $R(A)^\perp = N(A^T)$.

11) See the text.