MAS 3105 Dec 12, 2019
Final Exam + Key Prof. S. Hudson

1) Define $L : P_3 \to P_3$ by $L(p) = p'(x) - p(1)$. For example, $L(x^2) = 2x - 1$.
1a) [7 pts] Find a matrix representation of L for the ordered basis $S = \{1, x, x^2\}$ of P_3.

1b) [3 pts] Find ker L (Describe it with a simplified formula, or a spanning set, etc).

2a) [7 pts] Find 4 linearly independent eigenvectors of A. Label your answers clearly.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

2b) [3 pts] Can you find 4 eigenvectors of A that are mutually orthogonal? Do so, or show it is not possible.

3) [10 pts total] Let P_3 be the usual vector space of polynomials of degree less than 3. Let $S = \{p \in P_3 : p(4) = 0\}$. 3a) Show that S is closed under addition (it is actually a subspace of P_3, but you do not have to prove the other parts of the definition).

3b) Find a basis for S, and explain briefly, though you do not have to prove carefully that it is a basis.

4) Let A be a 4x4 matrix such that $\mathbf{a}_1 = [-3, 5, 2, 1]^T$ and $\mathbf{a}_2 = [4, -3, 7, -1]^T$.
4a) [7 pts] Find $\mathbf{a}_3$ and $\mathbf{a}_4$, given that the RREF of A is

$$U = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4b) [3 pts] Find a basis for the column space of A that does not include $\mathbf{a}_1$ or any scalar multiple of $\mathbf{a}_1$.

5) (8 pts) Show that if A is nonsingular, then AA^T is also nonsingular. Be sure to include enough words of explanation.

6) (8 pts) Compute and simplify e^{3A}, given that

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

7) (7 pts each) a) Is the following matrix diagonalizable ? Justify your answer. Hint: one idea is to look at the rank of $A - 2I$.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

7b) Find the eigenvalues of A^{-1} (possibly with repetition).

8) (20 points) Answer True or False. Assume the matrices are 3x3.

If U is unitary then $U + I$ is unitary.
If U is unitary then U^T is unitary.
If U is unitary then U^H is normal.
If T is a Hermitian upper triangular matrix, then T is a diagonal matrix.
A and A^T have the same eigenvalues.
If A is defective and similar to B, then B is defective.
If A and B have the same eigenvalues 4, 4 and 6, then they are similar.
If $\mathbf{x}$ is an eigenvector of A, then $2\mathbf{x}$ is an eigenvector of A^2.
If $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ is an eigenvector of $3A + I$.
If A is square and real then AA^T is diagonalizable.

9) (10pts) Choose ONE:

A) If $\{\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_k}\}$ are eigenvectors of A, from distinct eigenvalues, then they are L.I.

B) Prove that det $(AB) =$ det A det B. To save a little exam time, you can assume both matrices are nonsingular.

C) Suppose that A is an 8×5 matrix of rank 3 and let $\mathbf{b}$ be a non-zero vector in $N(A^T)$. Show that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Bonus: (5pt) This problem is based on Schur's Theorem and its proof, though you may find some other way to solve it. Find a unitary matrix W such that $W^H AW$ is upper triangular, given that

$$A\mathbf{w} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 2\mathbf{w}$$

Remarks and Answers: The average was 46 / 100 with high scores of 69 and 67, which is very low. The worst results were on problems 1 to 4, which were objectively not very hard, but maybe less familiar. The results were pretty normal on problems 5 to 9 which were more similar to problems from the homework or past exams. This may bring the previous scale down a few points, but I have not yet figured the effects of HW and MHW, which may raise it back.

1a) Write $L(1)$, $L(x)$ and $L(x^2)$ as column vectors to get

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

1b) Ker L = span $\{x\}$. You can get this from basic calculus + algebra. Or, from $N(A)$ =span$[0,1,0]^T$.

2a) The polynomial is $(1-\lambda)(2-\lambda)^2(3-\lambda)$ and the evecs are $[1,-1,0,0]^T$, $[1,1,0,0]^T$, $[0,0,1,0]^T$ and $[0,0,1,1]^T$. I got these mainly by guessing, but GE is standard. Other answers are possible.

2b) No. The ones in 2a are "almost orthogonal", but $\mathbf{x}_3 \perp \mathbf{x}_4$ is false. Success is not possible, because A is not normal (check that $A^TA \neq AA^T$).

3a) Suppose p, q are in S so $p(4) = q(4) = 0$. Then $(p+q)(4) = 0$ and $p + q \in P_3$, so we have additive closure.

3b) S = span $\{x-4, (x-4)^2\}$. I did not expect much justification, but if you are worried (for example) that the spanning set needs a third vector, please convince yourself that dim S = 2.

4a) Observe that $\mathbf{u}_3 = 2\mathbf{u}_1 + \mathbf{u}_2$. Since U and A have the same dependency relations, $\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2 = [-2,7,11,1]^T$. Likewise, $\mathbf{a}_4 = \mathbf{a}_1 + 4\mathbf{a}_2 = [13,-7,30,-3]^T$.

4b) Since $\{\mathbf{u}_2, \mathbf{u}_3\}$ is a basis for R(U), we know $\{\mathbf{a}_2, \mathbf{a}_3\}$ is a basis for R(A). There are many other answers, such as $\{\mathbf{a}_3, \mathbf{a}_4\}$, etc.

5) We know the determinant provides a test for singularity, so one easy proof is det (AA^T) = det A det A^T = (det $A)^2 \neq 0$. Another proof could be based "the product of two nonsingular matrices is nonsingular". Either proof deserves just a little more explanation.

6) Replace D with $3D$ and then

$$e^{3A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^3 & 0 \\ 0 & e^6 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^3 & e^6 - e^3 \\ 0 & e^6 \end{pmatrix}$$

7a) No. Since $A - 2I$ has rank =2, it has nullity =1, so we will only get one evec from $\lambda = 2$ (which has multiplicity 2).

7b) 1/2, 1/2, 1/3 from the $1/\lambda$'s. I did not expect more justification than this (see me if you want a proof).

8) FTTTT TFTTT. The seventh would be true if both were diagonalizable.

9) Parts A and B are in the text and the lectures. Nobody chose Part C, exercise 5.3.10.a. The fairly easy idea is that $\mathbf{b} \in N(A^T) \cap R(A) = \{\mathbf{0}\}$, a contradiction.

Most people chose B and followed the text and did OK. But many failed to mention a key idea, that det AE = det A det E whenever E is elementary. The textbook proof includes a comment that 'we have already seen that the result holds for elementary matrices' and you should too. My lecture proof is similar, but uses det EB = det E det B (with the E on the left) which is a little simpler, in my opinion.

Bo) $W = 10^{-1/2} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$. The first column is the given eigenvector $\mathbf{w}_1$ (based on the proof of Schur). The second column is orthogonal to $\mathbf{w}_1$, found by guessing (but GSO would also work), and the $10^{-1/2}$ is for normalization. If you trust Schur, you don't need to check that $W^H A W$ is upper triangular (but I checked).