The Rabbit Example

This example is pretty similar to one in the book about migration in and out of a city. Maybe it is slightly simpler. I usually give it in a lecture soon after covering Linear Transformations (see Leon, Ch 4) though some aspects make more sense later, with Ch 6.

We will study the growth of a population of rabbits, focusing only on the number females. We'll assume every adult female rabbit gives birth to 2 female babies each year. Each baby requires one year to become an adult. For simplicity, we will assume rabbits never die, and that the number of male rabbits has no effect on the story. We will start at time 0 with only one adult female, no babies. We will write this info in vector notation as

$$\mathbf{x}_0 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

According to our assumptions above, if the population vector this year is $[a, b]^T$, then next year it will be

$$\begin{pmatrix} a+b\\2a \end{pmatrix} = \begin{pmatrix} 1 & 1\\2 & 0 \end{pmatrix} \begin{pmatrix} a\\b \end{pmatrix} = A \begin{pmatrix} a\\b \end{pmatrix}$$

So, after one year, we have $\mathbf{x}_1 = [1, 2]^T$, then $\mathbf{x}_2 = [3, 2]^T$ then $\mathbf{x}_3 = [5, 6]^T$, etc. Note that in general $\mathbf{x}_n = A\mathbf{x}_{n-1} = A^n\mathbf{x}_0$.

Exercise: Compute the population vector in 100 years, \mathbf{x}_{100} .

Solution: The answer is $\mathbf{x}_{100} = A^{100}\mathbf{x}_0$, but this is too hard to compute directly. Let $\mathbf{v}_1 = [1, 1]^T$ and check that $A\mathbf{v}_1 = 2\mathbf{v}_1$. Do not worry about how I chose \mathbf{v}_1 , which we will get to later, in Ch6. But since

$$A\mathbf{v}_1 = 2\mathbf{v}_1$$

this vector is a very special and important one, called an eigenvector (and 2 is called an eigenvalue). Multiplying both sides by A repeatedly, we get

$$A^{100}\mathbf{v}_1 = 2^{100}\mathbf{v}_1$$

If we had been given $\mathbf{x}_0 = \mathbf{v}_1$ we would be done ! But that would be too lucky. We will come back to \mathbf{x}_0 soon. Let Let $\mathbf{v}_2 = [1, -2]^T$ and check that $A\mathbf{v}_2 = -\mathbf{v}_2$. This is another eigenvector, and we get

$$A^{100}\mathbf{v}_2 = \mathbf{v}_2$$

because 100 is even. Note that \mathbf{v}_1 and \mathbf{v}_2 are easier to handle than \mathbf{x}_0 , so we will write \mathbf{x}_0 as a linear combination of these two easy vectors. That means finding its coordinates wrt the basis $B = {\mathbf{v}_1, \mathbf{v}_2}$. I'll use the same letter *B* for the matrix we get from this basis (maybe not best, but fairly common) Using the transition matrix B^{-1}

$$\mathbf{x}_0 = \frac{1}{3} \begin{pmatrix} 2 & 1\\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1\\ 0 \end{pmatrix} = \begin{pmatrix} 2/3\\ 1/3 \end{pmatrix}_B = \frac{2}{3} \mathbf{v}_1 + \frac{1}{3} \mathbf{v}_2$$

The rest is easy. We multiply on both sides (ignoring the middle terms) by A^{100} and distribute

$$\mathbf{x}_{100} = A^{100}\mathbf{x}_0 = \frac{2}{3}A^{100}\mathbf{v}_1 + \frac{1}{3}A^{100}\mathbf{v}_2 = \frac{2}{3}2^{100}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2$$

and done. If desired, we can simplify to $\left[\frac{2^{101}+1}{3}, \frac{2^{101}-2}{3}\right]$, and observe that approximately half the rabbits are adults after 100 years.

Important! Try to relate this example to various topics we are learning this term. For example, A represents a linear transformation (in a Markov process). We needed a transition matrix, to change the basis of R^2 , from the standard one to an eigenvector one. This exercise can also be solved using ideas from Ch 4.3 and Ch 6. Here is an outline of that;

Alternative Solution: Check that the matrix representation of our linear transformation wrt B is

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

(it always happens this way - the diagonal entries are the eigenvalues). Based on Ch 4.3 ideas, D must be similar to A, so that $A = BDB^{-1}$. You can check this, of course, but this also has to happen. So,

$$A^{100} = [BDB^{-1}]^{100} = BD^{100}B^{-1}$$

because most of the B and B^{-1} cancel. The rest is easy, because powers of a diagonal matrix are easy. Compute D^2 for yourself and then you will probably agree that

$$D^{100} = \begin{pmatrix} 2^{100} & 0\\ 0 & 1 \end{pmatrix}$$

Now, compute $A^{100}\mathbf{x}_0$ using these formulas, to get the same answer we got before.