Linear Algebra, Examples of Proofs, mostly from HW 2
5/15/02

Ex 1 - True or False? If $AB = O$ then either $A = O$ or $B = O$.

Answer: False. **Proof:** We need an example (as in 1.3-20) and expect that A and B should be "almost zero". Some trial and error leads to these two, and it's easy to check they work.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex 2 [see 1.3 - 25] **Proof:** Assume that A and B are symmetric nxn matrices. The "if and only if" sentence has 2 separate parts that we handle one-at-a-time.

Part 1): Assume that $AB = BA$ (we'll prove AB is symmetric by showing $(AB)^T = AB$). So, $(AB)^T = B^T A^T = BA = AB$ (done). The first "=" is a theorem, the second is because A and B are symmetric, and the third is the hypothesis of Part 1.

Part 2): Assume that $(AB)^T = AB$ (we'll prove $AB = BA$). It's similar to Part 1): $AB = (AB)^T = B^T A^T = BA$ (done). [I'll let you explain each "=" this time].

Ex 3 [1.4-15] **Proof:** Assume B is singular. So, part a) of Thm 1.4.3 is false for B. So, part b) [and c)] is false too. So, $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Multiplying by A, we see that $AB\mathbf{x} = \mathbf{0}$ has the same nontrivial solution. This means Part b) of thm 1.4.3 is false for AB. So is a), so AB is singular.

Ex 4 [1.4 - 21a] **Proof:** The assumptions are that $A = E_k E_{k-1} \dots E_1 B$ and that $B = F_k F_{k-1} \dots F_1 C$, where the Es and Fs are all elementary.

[These equations are carefully based on the *definition* of "row equivalent". Don't try to write this proof without the Es. And the second equation has a new list of matrices, so use another letter, like my Fs, there. Back to the proof...]

So, $A = E_k E_{k-1} \dots E_1 F_k F_{k-1} \dots F_1 C$, which shows A is row equivalent to C. Done.

[1.4 - 21b]. By Thm 1.4.3, both are row equivalent to I. By Part a) above [and 1.4-20], they are row equivalent to each other.