Notes on HW Exercises
Summer A 2002

These hints and answers may help with some of your HW this semester, mostly HWs 3,4,5. The Student guide on reserve in the FIU library has much more. Try each problem pretty hard on your own before looking for hints. Most of the "proofs" below are just hints - so write them up more carefully, and fill in the gaps.

I have copied a lot of this from old (paper)handouts written for the 5th Edition, but have tried to update the exercise/page numbers. Also, I am not sure all of these were actually assigned this term - if not, I'd suggest trying them anyway.

1.4-15. Assume that B is singular. By a theorem in the book (1.4.3), it is false that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, there is a nonzero vector $\mathbf{x}$ so that $B\mathbf{x} = \mathbf{0}$. Multiplying both sides by A shows that $C\mathbf{x} = \mathbf{0}$, too. So it is false that this (second) system has only the trivial solution. So, C is singular (by the same theorem).

2.1-10 [also done in class]. [I'll omit the basis step (the 2x2 case) and the intro to the I-step.] Let A be an (n+1)x(n+1) matrix with two identical rows. For simplicity, let's assume they are at the top (otherwise, multiply by type I's to move them to the top). Compute det (A) using the 3rd row. det $(A) = a_{31}A_{31} + \ldots$, where $A_{31} =$ det M_{31} (etc). Notice that M_{31} is an nxn matrix, and its top two rows are equal (they come from the top two rows of A). By the IH, $A_{31} =$ det $M_{31} = 0$ (likewise, for A_{32}, etc). So, det A = 0.

2.2-13. Taking det's, we see that det A is not zero, so A is nonsingular, and we can use A^{-1} in the rest of the proof. We multiply it by $AB = I$ to get $B = A^{-1}$. We know $A^{-1}A = I$ (by defn), so $BA = I$.

2.2-17. Finding the det of a very large matrix (say 100x100) can take some time, even with a computer. A practical question is to choose a method that takes relatively few steps. This problem shows A LOT of steps are required in our main method, using cofactors - because 100! is BIG.

Expect to use induction, since this involves det's. I'll only do the addition part here, since the other part is similar.

Basis step: A 1x1 matrix requires 0 additions, which agrees with the formula $n! - 1$ when $n = 1$.

Induction step: Let $k \geq 1$ and assume the formula is correct for kxk matrices. Let A be a (k+1)x(k+1) matrix. Then using row 1,

$$detA = a_{11}A_{11} + \ldots + a_{1k+1}A_{1k+1}$$

which has k additions in it. But finding each cofactor (such as A_{11}) also requires $k! - 1$ additions (by the I.H.). So, the total is $k + (k+1)(k! - 1)$. This equals $(k+1)! - 1$, so we're done.

Remark: I suppose you *could* also count the plus signs that occur in "$(-1)^{i+j}$" (in the definition of A_{ij}). But this is apparently not the author's intention (play around with some specific examples, as a check, before you start a long proof).

If you didn't get the multiplication part yet, try again!

3.3 - 12. Given $a_1\mathbf{v_1} + a_2\mathbf{v_2} = \mathbf{0}$; solve for $\mathbf{v_1}$ [But what if $a_1 = 0$? Also, reverse this logic for "part 2" of the proof].

3.3 - 13. If a nontrivial combo of the subset were zero, a similar non-trivial combo (all new coeff's are zeros) of the big set would be zero. [try to write this idea out with formulas].

3.3 - 16. Since $\mathbf{v}$ is a lin combo of the others, we get $\mathbf{v} = a_1\mathbf{v_1} + \ldots a_n\mathbf{v_n}$. Bring the $\mathbf{v}$ over to the right, and the new eqn shows $\mathbf{0}$ is a nontrivial combo of the vectors.

3.4 - 11. This is like the nullspace examples with α and β in them (see pg. 119). Factor out the a and b and get $a(x^2 + 2) + b(x + 3)$. So each element of S is a lin combo of $x^2 + 2$ and $x + 3$.

3.5.9a) (read ex.7 first) The first column comes from the coefficients in $2x - 1$, so it is $(2, -1)^T$. The second column is $(2, 1)^T$. [If the other basis had been written "$[1, x]$", then the entries of col 1 would be reversed, to $(-1, 2)^T$, and the same for col 2].

3.5.9b) Take the inverse of the previous answer.

3.6.9a) idea: the columns have the same dependencies in A and in B. or faster, you could use thm 3.6.5, but that's almost cheating).

3.6.9b) No (see lecture ex. (from year 2000 - and 2002 also?), where col.1 of A is not in Col(U)).

3.6.19b) See the proof of Thm 3.6.1. 19a) can be explained almost the same way, or we can think of the columns of C as the rows of its transpose. The notation isn't perfect, but basicly: Col(AB) = Row $((AB)^T)$ = Row $(B^T A^T)$ is a subspace of Row (A^T) = Col(A). 19c) From (b) we see rank(C) is at most rank(B). By (a) and by Thm. 3.6.5, we see rank(C) is at most rank(A). Done.

4.1.20) If L is one-to-one and $L(0) = 0$ (given), then $L(x) = 0$ can't happen for any $x \neq 0$. So, Ker(L) contains only 0. Conversely, suppose L weren't one-to-one. So, $L(\mathbf{v_1}) = L(\mathbf{v_2})$. So, $L(\mathbf{v_1} - \mathbf{v_2}) = 0$. So, $\mathbf{v_1} - \mathbf{v_2} \neq 0$ is in Ker(L).

4.3-9) Get $S^{-1}A = T$ and plug in to get $B = S^{-1}AS$.

4.3-11) det B = det S^{-1} det A det S = det A. (see thm 2.2.3 and HW 2.2-6).

5.1-9) Find the scalar projection of $[1, 1, 1]^T$ on $[2, 2, 1]^T$ as in Example 5.

5.2-13a) Ax is in R(A) by definition. Since $A^T(Ax) = 0$, it is also in $N(A^T)$. Since these are orthogonal subspaces (thm 5.2.1), this implies $Ax = 0$ (remark 1, on page 241). So $x \in N(A)$. This gives 13b).

13c) By 13b), they have the same nullity. They also have the same width, so by thm 3.6.4, they have the same rank.

13d) The assumption implies rank(A) = its width. By 13c), rank ($A^T A$) = its width. For a square matrix, this implies nonsingular (see Cor.3.6.3).