Induction Proofs and Determinants

I have tried to show the basic idea of recursive definitions and induction proofs in the lectures. If you want to read more on the basic idea, try a *Discrete Math* textbook, perhaps from the library. The end of this web page also has a short summary. But here I will mainly discuss one example of an induction proof about determinants in great detail.

Example 2.1.2: Look at Thm 2.1.2 on page 105, which says $\det(A) = \det(A^T)$ (for any $n \times n$ matrix A). The textbook proof is completely correct and convincing, but the author does not try to show where it came from, and that is not really the writer's job. So, I will guide you through the discovery process for this proof.

Let's look at a specific matrix A. We may already be pretty sure the statement is correct, but we are going to check it anyway, hoping to learn the main idea - the hidden connection between the smaller cases and the larger ones. Of course, this matrix will not appear in our final proof. Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

To compute $\det(A)$, use column 1 (since it has 2 zeroes), and get $\det(A) = 0 + a_{21}A_{21} + 0 = -2\det(M_{21}) = -2(-4) = 8$. We want to compute $\det(A^T)$ *in a similar way*. Use row 1, and get $\det(A^T) = -2\det(N_{21}) = -2(-4) = 8$. Here I'm using M with A and N with A^T:

$$M_{12} = \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} \qquad N_{21} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} = M_{12}^T$$

These calculations show that $\det A = \det A^T$, but mainly they help us find reasons this should always be so. Let's summarize:

a) The entries in column 1 of A match the entries in row 1 of A^T.
b) The minors also match: $\det N_{21} = \det M_{12}$
c) The reason for b) is that $N_{21} = M_{12}^T$ and
d) $\det M_{12}^T = \det M_{12}$

It seems pretty clear that a), b) and c) will always be true. But can we be sure about d) ? Part d) is pretty clear for 2x2 matrices, but not for larger ones. In fact, statement d) is one case of the theorem we are aiming to prove! This is the hidden connection we need - the 3x3 case is true *if* the 2x2 case is true. Note that the actual entries in A don't matter. Also, if A is 4x4, the calculations work the same way, except that the minors involve 3x3's.

Now that we have the right idea, writing down a general induction proof of Thm 2.1.2 should be pretty routine (but it takes some practice, of course). Every induction proof has two parts; **the basis step** (usually very short) and **the induction step**. The basis step handles the smallest case. Here, that's when A is a 1x1 matrix, $A = (a)$. Since $A = A^T$, the theorem is pretty obvious in this case. [Note: Theorem 2.1.4(ii), for example, doesn't make sense for 1x1 matrices, so the basis step should be about 2x2's].

The induction step begins with sentence 3 of the author's proof, "Assume that the result holds for all $k \times k$ matrices, and that A is a $(k+1) \times (k+1)$ matrix". This *same sentence* can be used in almost any induction proof about square matrices (eg in your Ch. 2 H.W.). In our practice example k was 2, but now we are aiming for a more general pattern, and won't assign a number to k. The rest of the proof ("Expanding ...") just explains the idea we found in the example.

Summary: Induction proofs usually have an easy basis step and a pretty standard third sentence. The rest will vary from proof to proof, and should explain the hidden connection between larger and smaller sized matrices. You can usually find this connection by patiently playing around with 3x3 examples.

For even more about this same proof, see my proof pages (6-8) from Summer 2000. But if you followed everything above, you can probably skip that, as well as the rest of this page.

Review of the Basics; Recursion and Induction

Recursion: The definition of determinant includes matrices of size 1x1, 2x2, 3x3, etc. So, it has to include a lot of cases at once. It's easiest to do this by relating each case to a previous case. We call such a definition "recursive".

For a simpler recursive example, define $n!$ as $n(n-1)!$. This implies that to find 7!, we would need to find 6! first (and so on, like the reduction formulas you learned in Calculus). We may have to repeat this many times, but eventually the problem reduces to knowing that $0! = 1$ (this is often called the "basis" part of the definition of $n!$).

The definition of determinant works like that, too. The 3x3 case uses the 2x2 case, etc.

Induction: Proofs that rely on this kind of definition, usually use the "induction" method. This is a way to prove a statement that includes a lot of cases (often indexed by the letter "n") all at once. A statement like "$n! > 2^n$ (for all $n \geq 4$)" or a statement about all nxn matrices.

It uses the smaller cases to prove the larger cases. For example, if you believe that $4! > 2^4$ then you can multiply the left side by 5, and the right side by 2, to see that $5! > 2^5$. That is the idea behind the proof that $n! > 2^n$, but to write it correctly, you need to include a basis step (checking that $4! > 2^4$) and an induction step (using the previous idea to show that the pattern continues forever). You can find simple examples involving $n!$ (etc) in any Discrete Math book, or in the book by Morash, on reserve in the library.

Most proofs about determinants require induction. Before you can write such a proof, you need to find the connection between the small cases and the large cases. Sometimes you can get the right idea quickly by looking at well-chosen examples, and sometimes I will prefer that in my lectures over a careful proof. But I'd like you to be able to do a few induction proofs, such as for Theorems 2.1.2, 2.1.3 and 2.1.4.