Linear Algebra Supplement, Summer 2000
Sections 1, 2, 3

Answer the 10 questions in this handout as part of HW 1. This should help you get started in the course, especially if this is your first course involving proofs. Most Linear Algebra textbooks assume that you have some experience at writing proofs. If you don't, this handout may help. You might also try:

a) *Bridge to Abstract Mathematics* by Morash, on reserve in the FIU library. Also, many bookstores have good books about proofs.

b) Try to prove examples that are worked out in the text book (for example, see pg 22, pg. 50, or pg 52). These will probably be hard at first. See Schaum's or other texts for more examples.

c) The HW will gradually include more proofs. Feel free to ask me for hints and/or criticism, either before or after it is due.

Section I. Why learn proofs?

Scientists of all types search for the truth. They look for patterns, then for confirmation of their ideas. They might use experiments, surveys or debates. But in mathematics we must rely on pure reasoning - proofs.

A proof is a *precise* explanation of why a statement is true or false. A conjecture that has been proven is called a theorem, and can be used in other proofs. So, theorems build on each other like bricks in a tower. One little mistake in reasoning ruins the whole thing, so we should be pretty careful. Each theorem must be stated very precisely, to be really useful later on.

You will have to write some proofs to succeed in this course; maybe one per exam. "Understanding the material" is not enough - you'll need to use *logic* and *definitions* in each one. You can start now, learning the exact definitions of words like *consistent, symmetric, nonsingular* etc. that appear in Chs 1-2; you'll need these. With experience, you may find that some words, like (linear equation, Gaussian elimination, pivot, etc) aren't used much in proofs and just "getting the right idea" is OK.

These handouts will show you how to use logic (and definitions) to start your proofs. You'll be tempted to use examples, experience and intuition to start your proofs, but don't rely too much on these. A good proof never appeals to the reader's intuition.

Question 1: Name a few qualities (mentioned above) of a good proof. (clear? witty? logical? interesting? precise? brief? intuitive? organized?)

Question 2: Name 2-3 goals of these handouts; that is, name 2-3 tools you'll use to write proofs.

Section II. An important little word; "AND"

True or False: The PC building is big and beautiful.
True or False: The PC building is big or is beautiful.

You probably sorted out the difference between the two sentences without thinking very much about the meaning of AND and OR. But how do they affect our answer? What if we had to justify (prove) our answer?

Most statements you'll meet in Linear Algebra will consist of several simple phrases, connected by little words like "and", "if", "or" etc. These little words determine your plan for a proof of the statement. Understanding these very very well is your first step, and is the main goal of these handouts.

Suppose you have to prove that a certain system is "homogeneous AND underdetermined". (it's OK not to know the meanings yet). Then you would do the proof in two steps; 1) prove it is homogeneous and 2) prove it is underdetermined (don't worry about how to do those steps yet). In general, to prove a statement of the form "p and q", you split the job into two independent parts [and the reasoning used in part 1 is probably not relevant to part 2].

Complications: Some "AND statements" have more than two parts, and sometimes the "and" is not written out. For example, look at thm. 1.3.1, pg.42. This statement is really an "AND sentence" with 9 parts. So, the proof has 9 separate steps (only 2 are shown in the textbook). See also Ch. 1.3 exercises 16, 17 and 24. For practice, scan the text for more theorems and homework that are "and" statements. See also the definitions on pgs. 111, 116, etc.

Statements of the form "p if and only if q" are common (see the corollary on page 61, and HW 1.3 - 24, for example). This means the same as "if p then q *and* if q then p". The proof will have two steps (explained later).

Sometimes, we will want to show that "p and q" is *false*. This does not require two steps - just one. You just have to decide which phrase (p or q) is false, and prove that your decision is correct. For example, "The PC building is big and beautiful" is proven false if you can prove it is not beautiful.

See ex. 3, pg. 15, remembering that the definition above it says "(i) *and* (ii) *and* (iii)". The proof for the first matrix takes only the one sentence; the author just mentions that (i) is false (which he considers obvious).

Question 3: Look at problem 3 on page 114, and at the definition of vector space on page 111 (you don't have to understand it yet). How many steps would it take to show that C is a vector space?

Question 4: How many steps would it take to show that a system is in reduced row echelon form? (use the index to find the definition, if necessary)

Question 5: Find four "and" -type definitions in our text.

Question 6: Find two theorems (or corollaries) of this type.

Now you should know how to recognize an "and" statement, even in disguise, and should be able to outline a proof of one. You should also realize that a statement of the form "p or q" has an entirely different meaning than "p and q", and calls for a completely different plan. The next sections cover 5 more little words, like "or", "if-then", etc. When you can handle these little words correctly, then you have the logic that you need to write proofs in a mathematics course.

Section III. "If-Then"

Almost every theorem has a hypothesis. So, it is of the form "if p then q", where "p" is the hypothesis and "q"is the conclusion. For example, in Thm.1.2.1 (pg 22) the hypothesis is that a system is homogeneous and underdetermined. The conclusion is that it has a nontrivial solution. You normally begin to prove such a statement by assuming that p is true and then you prove q.

For example, I might begin the proof on page 58 with the sentence "Assume that A and B are nonsingular n x n matrices." Then I would write, "We will prove that AB is also nonsingular and ..." before beginning the real work. (Note that "q" is an "and" phrase so that Lesson 2 would take over at this point). The author often takes shortcuts, which I don't recommend to you (yet).

There are other ways to write "if p then q", including "p implies q", "q if p", "p only if q", "$p \to q$, and "q whenever p". This can be confusing. The *converse* of "if p then q" is "if q then p" and has a completely different meaning.

Try to interpret exercise 9b on page 14, and to begin a proof. Hint: it means the same as "if $m_1 = m_2$ and the system is consistent, then $b_1 = b_2$."

True or False: If X is a four-legged animal then X has fur (False, Why?). How do you show that an "if p then q" statement is *false*? You find an example for which "p" is true, but "q" is false. (for question above, a lizard is a good example). For practice - is it true that "If AB is the zero matrix, then either A or B is the zero matrix" ? Hint: exercise 19 on pg. 55 asks for a relevant example.

Question 7: Set up a proof of 9a and then 9b, pg. 13 by stating your assumptions and goals.

Question 8: The proof of 24 on pg. 56 will have two parts, because it's an "and" sentence. State what the parts are and what assumption(s) you'd make in each part. (You'll do the whole proof in HW2).

Question 9: Suppose you believed Thm. 1.2.1 to be false. How would you try to prove it false? (just give a plan).

Question 10: I would appreciate any constructive criticism of this handout. Has it helped you? confused you? (if so, which parts?) bored you?

Other suggestions: if you'd like to get ahead, you might start thinking about these (optional):

1) Read over example 2, page 283, to see that "eigenvalue" isn't a very hard concept.

2) Draw a picture of the vectors $w = (3,1)^T$ and $v = (1,4)^T$ and find a vector parallel to w whose tip is as close as possible to the tip of v. You could try calculus and/or geometry. (example 3, pg 200, gives a fancy solution).

3) Referring to the same v and w from 2), notice that these two vectors are not perpendicular (not *orthogonal*). Can you describe the set of all vectors that *are* orthogonal to v?