Linear Algebra, Summer 2000
Supplements 4-5

Part 4. Definitions.

In daily life, we rely more on intuition than on precise definitions. But in some situations, like mathematics, legal arguments, games, programming, etc definitions and rules are more important. For example:

Is it *hot* outside?

The answer is just a matter of opinion, unless *hot* has been defined. If such a question were important to us in a courtroom, or a math course, we would probably introduce a definition, to make the question precise, such as:

Defn: *Hot* means above 80 degrees Fahrenheit.

For a math example, look at Thm. 1.4.1 on page 58. The goal is to show AB is nonsingular. The proof is based on the definition of *nonsingular* on page 48. (the "A" on page 48 is replaced by "AB" and the "B" on pg 48 is replaced by "$B^{-1}A^{-1}$"). You could not possibly write this proof yourself without knowing this definition. In fact, if you don't know how to start a homework proof, you might begin by studying the related definitions.

Next, look at exercise 24 in Ch.1.3 (from HW 1). You should already know how to split it into 2 parts (it's an AND sentence) and how to start each part (both parts are IF sentences):

Part 1. Assume that A and B are symmetric (so $A^T = A$ and $B^T = B$). Assume that $AB = BA$. We must show that AB is symmetric.

What's the next step? Usually, it's best to focus your attention on the desired conclusion (that AB is symmetric) and look for a way logic or a definition. In this case, the conclusion contains no little logical words like "and", "if", etc. but it does contain a technical word "symmetric". So, we should look at the definition of that word (pg. 50) to decide what the conclusion really says. We replace the "A" on pg. 50 by "AB" and see that our goal in Step 1 is really:

$$(AB)^T = AB$$

That's good - equations are usually easy to prove. Just calculate $(AB)^T$ (use the hypotheses $A^T = A$ and $B^T = B$ and a 'Rule' from pg 50 - include justifications).

Part 2 of the proof is similar (see question 8 of the previous lesson, and also the next section about proving equations).

The main point of this lesson is to depend on exact definitions (of words like *nonsingular*, or *symmetric*) when writing proofs. Don't depend too much on your intuition, or you will often find yourself stuck with nothing to say, or may find your proofs graded "unclear" or "imprecise".

Here is some practice with logic and definitions - you will not be able to use intuition in this exercise because it involves nonsense words!

The story: In Hudsonville, there are three kinds of people - called neds, mils and pips. No one is two types. Also, each person is either red, yellow or blue. You are given 4 definitions:

1) A *donul* is anyone who is a ned or a mil.
2) Anyone who is not a mil is called a *goordo*.
3) A red pip is called a *gingar*.
4) A *felix* is a red goordo.

Example: Show that every gingar is a felix.

Solution: Let g be a gingar. So g is red, by definition 3. Since g is a gingar, g is a pip, so g is not a mil. So, g is a goordo, by definition 2. Since g is a red goordo, g is a felix (definition 4). We have shown that an arbitrary gingar g must be a felix.

Question 1: Show that every felix is a gingar or a donul.

Question 2: Look at problems 1.3 - 9, 22, and 1.4 - 18, 19. Which definitions would you need for each problem?

Question 3: Many definitions involve little logic-words like "and", "if", "or", "exists", "for all" and "not". They may be explicit or understood. Which ones are in the definitions of 'triangular form', 'row echelon form', 'equal' (pg 32), 'nonsingular', 'singular' and of 'donul' and 'gingar' above?

Part 5. Proving Equations, and answering True-False.

Equations are relatively easy to prove - the proof is often just a calculation (remember trig identities?). Some are of the form "number=number" (like 2.1 - 9) or of the form "matrix=matrix" (like 1.3 - 24, discussed above). Here are some standard approaches for these:

1) Start on one side (probably the more complicated side) and calculate away until it simplifies down to the other side.

Example: That's a good way to do 1.3 - 24. ($(AB)^T = B^T A^T = BA = AB$) Justify each step.

Example: The main idea of 2.1 - 9 is to calculate det (A), using theorem 2.1.1.

Example: For a variation of this idea, see the proof on page 42, where both sides are calculated separately and are found to be equal.

2) Start with a known formula, one that's similar to the one you want to prove, and "do something" to both sides. This "something" should make the formula more like the one you want. If the given problem involves A^{-1}, you can probably begin your calculations with the known formula $A^{-1}A = I$.

Example: problem 2.2 - 6 could start this way. What would you do to both sides? Since it involves determinants, you should probably take determinants of both sides of $A^{-1}A = I$ to see what happens. (Use theorem 2.2.2 to do this).

Example: In another problem, like 1.3 - 17, you might transpose both sides instead.

3) Use the definition of one side of the equation. For example, see 1.4 - 8a. By definition, A^{-1} is the matrix that makes $AA^{-1} = I$ and $A^{-1}A = I$. So for this problem, you just have to multiply the two matrices (both ways) and check that you get the identity matrix both times. Actually, according to 2.2 - 13, you can get away with just *one* multiplication. Problem 1.3-17 could also be done this way.

Other Equations: You may also see equations of the form "set 1= set 2", which usually require a different approach (because you usually can't "calculate" a set). for example, in ch.3, we'll discuss sets of vectors (vector spaces). The best approach is usually to show

a) that set 1 is contained in set 2 (Assume x is in set 1. Prove x is in set 2.) - and then

b) that set 2 is contained in set 1.

Part 5B: Answering True-False.

Anyone who has worked with mathematics has, at some point wondered, "Is it OK to use that formula here?" or "Can I go from this step to that step?" These are basicly True-False type questions. Normally we want an answer *and* some justification. Many exercises in the text, such as 2.1 - 11, are written in this form. One good approach is to guess "false" at first and look for a counterexample. If it really is false, this shouldn't be hard to do. But if your first few examples do not contradict the statement, you should change your guess to "true" and try to find a general proof (without using specific examples, usually).

Example: Suppose we know $AB = AC$. Is $B = C$ always true?

Solution: Guess "No" and try some examples. We'd like $AB = AC$ to be true, but for $B = C$ to be false. Consider the example $A = I$, $B = I$, and $C = O$ (the zero matrix). This makes B different from C (as we want), but unfortunately AB is also different from AC, so this is not a counterexample. We try again. Let $A = O$, $B = I$ and $C = O$. Now B is different from C, but $AB = AC$. So we have a counterexample, and the answer is "No".

You probably answered 1.3 - 20 in a similar fashion, probably with actual numbers involved (which is fine). But most proofs of "Yes" *won't* use specific examples. Next time, I'll return to the little words of logic, and say more about when examples are OK.

Also, next time: a proof method called "induction". You'll need this for 2.2-16 (in HW 2). To get started, set n = 2, write down a 2x2 matrix, and find its determinant. Count how many times you added (include subtractions, too). One? That agrees with n!-1 = 2!-1 = 2-1 =1. Also, check this works for n = 3 and n = 4. You should see a pattern - notice that *the counting you do when $n = 4$ can be simplified using your answer for the case $n = 3$.* Induction is a precise way to summarize a pattern like this. See the proof of thm 2.1.2, for the proper style to use. (you might also consult another book, such as a Discrete Math textbook, or the book on reserve).