Linear Algebra, Summer A, 2000
Supplements 6,7 and 8; S.Hudson

This is the last handout on proof writing, and is intended to cover a few more "little words" of logic, and also induction. For more help, you can look at "Bridge to Abstract Mathematics" on reserve in our library, or you can come by my office. For more on induction, you might also look in a Discrete Math textbook.

Summary of Sections 1-5: When writing a proof:

a) Use complete sentences (beginning with a capitalized word - not with an equation).
b) Know the precise definitions before you start [also look for related theorems or exercises].
c) Know what kind of sentence you've got (AND ?, IF-THEN ? , etc) and which proof strategy goes with it.

These handouts deal mainly with c) - how to handle the usual little logical words and phrases; especially "and", "if-then", "or", "there is a", and "for all". The last three phrases are discussed below. For each of these 5 phrases, you should know

a) how to prove a statement of that type (how to begin, at least)
b) how to disprove such a statement
c) how to use such a statement if it is given as an assumption.

That's enough logic for this course. When you get used to this logic, proofs will become easier to set up.

I haven't talked much about proof writing *style*. But pretend that you are explaining something fairly technical to a skeptical little brother (who knows less than you) and your style should take care of itself. Mainly, be sure to enclose your formulas in complete sentences by using words like "Assume" or "We'll show that" or "So". Put your formulas into context!

Section 6. More little words; $\exists$ and $\forall$

6A. The phrase "there is a ..." appears so often in mathematics, it has an abbreviation, "$\exists$". For example, the definition of A *is nonsingular* could be summarized as $\exists A^{-1}$. The definition of *row equivalent* is also a $\exists$

type sentence (confirm!). We prove such sentences by giving a formula for whatever is supposed to exist.

Example: Prove there is a number $x > 0$ so that $x^2 < 5$.

Proof: Let $x = 1$. Note that $x > 0$ and $x^2 < 5$.

To prove A is nonsingular, we usually have to give a formula for the inverse (for example, see thm. 1.4.1 - a formula is given for $(AB)^{-1}$, and it is justified). Also, see exercise 1.4 - 20a: you have to provide a sequence of elementary matrices that show A is row equivalent to C (the hypotheses help you do that).

6B. The phrase "for all" is written "$\forall$". For example, we've seen that $\forall \mathbf{x},\ I\mathbf{x} = \mathbf{x}$. Likewise, the commutative property (see axiom 1 on pg. 111) could be written "$\forall x, y \in V, x + y = y + x$". Thm 1.3.1 is also of this type because it applies to all matrices.

This type of sentence can be handled like an if-then sentence. For example, the axiom above could be rephrased "If $x, y \in V$ then $x + y = y + x$". So, a proof would begin "Assume that $x, y \in V$. We'll show that $x + y = y + x$". (Don't worry about the rest for now).

These two phrases are called "quantifiers" because they put variables into context. They are opposites of each other, because to prove a $\forall$ sentence is false, we have to prove a $\exists$ sentence, and vice-versa.

For example, to disprove "There's a Catholic in the class" we'd have to show that "Every student in the class is not Catholic", which is a $\forall$ sentence. (Recall that "and" and "or" are also opposites).

Example: Show that "$\exists x > 0, x < -1$" is false.

Proof: This means the same as $\forall x > 0, x \geq -1$. So, assume $x > 0$. Since $0 > -1$, we get $x > -1$ (transitive property), so $x \geq -1$.

Example: Show that "For all matrices A and B, AB=BA" is false.

Proof: We must prove the opposite sentence (the *negation*), which is the $\exists$ sentence, "There are a pair of matrices A and B such that AB is not equal to BA". I'll let you find such a pair (it's easy - but the ones in 2.1 (2ab), pg 90 should work).

Question 1: Look at the definition on pg 38. What kind of sentence is the claim that "$\mathbf{v}$ is a linear combination of the vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots \mathbf{a_n}$"? Could one prove it by finding values of $c_1, c_2, \ldots c_n$?

Question 2: Look back at 1.3 - 19 (but you don't have to solve it again).

Answer each of these True or False.

a) $\exists A \neq O$ and $B \neq O$ so that $AB = O$.

b) $\forall A \neq O$, $B \neq O$, we have $AB = O$.

c) $\forall A \neq O$, $B \neq O$, we have $AB \neq O$.

d) If $AB = O$, then $A = O$ or $B = O$.

e) If $A = O$ or $B = O$, then $AB = O$.

f) If $AB \neq O$, then $A \neq O$ and $B \neq O$.

Which of the statements above are logically the same, and would have identical proofs (or dis-proofs)?

Section 7. "Or".

We sometimes prove a "p or q" sentence by choosing one part and proving it. For example, to prove that "I can jump or I can fly" I might simply jump, ignoring the part about flying. Usually it's harder than that in a math problem (fortunately, "or" doesn't appear too often). Often we can prove "p or q" by proving "if p is false, then q is true" which means the same thing, but is phrased in the more familiar "if-then" format.

Example: Show that for all real x, either $x > 0$ or $x < 2$.

Proof: Let x be an arbitrary real number. We'll prove that if $x > 0$ is false, then $x < 2$ is true. So, assume $x \leq 0$. Then (by transitivity), $x < 2$. This proves the "or " sentence.

Example: Look back to exercise 1 from Part 4 (about the 'felix'). One strategy is: assume the felix is not a gingar, and prove it must be a donul. [It is OK to use cases instead - see below].

Example, exercise 3.2 - 16 will be assigned soon. It is of the "or" type. You can begin "Assume S is a subspace of R^1 and that S is not $\{0\}$ (so $\exists x \neq 0$, $x \in S$). We'll prove that $S = R$." Etc. As a further hint on this problem, recall that to show two sets, like S and R, are equal, we usually show containment both ways. One direction ($S \subseteq R$) is obvious here, so you should focus on showing $R \subseteq S$; that is, show that every real number "y" belongs to S.

Now suppose that the "or" is in the hypothesis rather than the conclusion, so that we are *using* it instead of *proving* it. Here we may use cases. For example, the assumption "f is a felix" (see question 1 in Section 4) leads us eventually to two cases "f is a ned" or "f is a pip" (it can't be a mil), which we handle separately.

Sometimes the "or" in the assumptions is hidden. For example, Ch.1.1-9a says that m_1 is constant, but we don't know whether it is zero or not (and it matters in the proof). So, these two cases must be handled separately.

For another example, see Thm. 2.2.3. The proof is based on the fact that B has to be either singular *or* nonsingular. So, it consists of two separate arguments - one for each case.

How do we know when to use cases in such proofs? Normally, this kind of case-work is used as a last resort. You might get stuck in some proof, and say to yourself "Well, if I could assume statement p, I'd know what to do next". So, assume it! (and label it "Case 1"). Of course, you have to come back later and do Case 2 (achieving the same goal when p is false) and this would probably require some new ideas. But at least you wouldn't stay stuck!

Proofs involving elementary matrices often have to be split into cases depending on whether the matrix is type I, II or III. See the proof of Thm 1.4.2.

Question 3: Find a way to set-up problem 3.2 - 16, different than the way I suggested above. Base the proof on the fact that either $1 \in S$ or it's not in S. Write down the cases and go as far as you can.

Question 4: How would you prove that a statement of the form "p or q" is false? For example, show that the matrix below is *not* elementary (not type I or II or III). You can omit details here.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Section 8. *Induction.*

Sometimes a situation must be split into an *infinite* sequence of cases, and a new method, called *induction* is required. For example, if A is a square matrix, then it is either 1x1 or 2x2 or 3x3, etc., etc. If we are proving a statement about det(A), we are usually required to consider these different cases separately (because det A is defined recursively). How can we cover an infinite sequence of cases at once? We have to establish a pattern, such as:

a) Check the 2x2 case; this is called the *basis* step. [but sometimes we start with 1x1 instead].

b) Show that the 3x3 case is true *because of the 2x2 case.* (if we don't *need* to refer to the 2x2 case here, we probably don't need to use induction at all). You may look at specific examples to get the right idea here.

c) Show the 4x4 case is true because the 3x3 case is.

d) Show that the pattern begun in b) and c) continues forever; this is called the *inductive* step of the proof.

Actually, steps b) and c) are not necessary, and are usually not included in the final proof. See the proof of thm 2.1.2 (pg 89), for example. But if you are stuck on a problem, doing steps b and c should help.

Try to prove thm 2.1.3 now. If you get stuck, you might play around with examples like those on pg. 63 (trying to see the connection between 2x2's and 3x3's).

I did a few examples of this process in class, and (almost) did the following problem for you, before proving Thm 2.1.2.

Question 5: Do step c) above (for thm 2.1.2), using the 4x4 matrix on pg. 89. That is, compute $\det(A)$ using row 1, and $\det(A^T)$ using column 1 of A^T, and explain why the answers are the same. Try to explain this with a_{ij} and M_{ij} notation. [Your answer should involve equations very similar to those on page 89-90.]

Hopefully, Question 5 wasn't too hard, and by answering it you can see where the proof of Thm 2.1.2 comes from.

This is probably the last of my handouts on proofs. If you need help understanding any of them, or feel that more would help, please let me know.