Some Linear Algebra Proofs
Prof. S. Hudson

Here are some sample Linear Algebra proofs, mainly chosen from unassigned exercises in Chs 4.1-5.1 of the Leon textbook (7th Ed). Since there are already lots of model proofs in the text, I'll focus on the thinking behind these proofs here. I'll type my background thoughts in italics (these would not appear in the actual proof).

For more about proof strategies, I'd suggest reading Velleman's book. Besides that, my main advice is to focus on definitions and logic (also, previous theorems are often useful). By 'logic' I mainly mean respecting the basic sentence structure of the claim you're proving. For example, the claim that *S is a subspace* is a sentence of the form *1 and 2 and 3 and 4 are all true* (see the def from my lectures). So, your proof will have 4 separate parts; 4 small proofs, really. Also, pause a minute when you are done with each proof, to review it; check for gaps or weak explanations.

To benefit from these exercises, try them yourself first, at least 5 minutes each, before you read my answers. If stuck, you might just read my first few lines, and then see how much you can do by yourself.

Ch 4.1.20: [Show that $L^{-1}(T)$ is a ssp of V]

Hmmm. A tough one? Since I can't connect this to any recent theorems, I'll probably need to understand and use the def's of subspace and of inverse image (pre-image). As a math prof, I'm very comfortable with that. But if I were still a student, I'd have to review those definitions until they made good sense. I'd also look over Theorem 4.1.1, which looks very similar to this exercise. I don't see any way to use that theorem directly, to prove exercise 20, but probably the rough outlines of the two proofs will be similar. Well, I should focus on the definition of subspace *first, and worry about pre-image later. Let's start with closure under multiplication, and see how that goes.*

Let $\mathbf{x} \in L^{-1}(T)$, and let α be some scalar. We need to show that $\alpha\mathbf{x} \in L^{-1}(T)$. *Hmmm. Gotta deal with the L^{-1} now. Twice actually! Once for the hypothesis, and once for the conclusion. After some thought, I realize that $\mathbf{x}$ is one of the $\mathbf{v}$'s in that def, and I want $\alpha\mathbf{x}$ to be one, too. That's not very precise language, but I can phrase it better:*

Since $\mathbf{x} \in L^{-1}(T)$, we know $L(\mathbf{x}) \in T$. Since T is assumed to be a subspace of W, we know $\alpha L(\mathbf{x}) \in T$. Since L is linear, this means the same as $L(\alpha\mathbf{x}) \in T$. But by def, that means $\alpha\mathbf{x} \in L^{-1}(T)$.

The last few lines were fairly easy, since I've seen similar manipulations in other proofs, like Thm 4.1.1. I will leave the rest to you. Closure under addition should be very similar to this. Ideally, you should also show that $\mathbf{0} \in L^{-1}(T)$ (so, it is non-empty) and that $L^{-1}(T) \subseteq V$ (thus including all four parts of the def of subspace). Both of these are fairly easy, once you know how to use the def. In fact, Leon didn't even bother to mention the $\subseteq$ part when proving Theorem 4.1.1. Here's the proof of the first closure property, without *my thinking mixed-in:*

Just the proof: Let $\mathbf{x} \in L^{-1}(T)$, and let α be some scalar. We need to show that $\alpha\mathbf{x} \in L^{-1}(T)$. Since $\mathbf{x} \in L^{-1}(T)$, we know $L(\mathbf{x}) \in T$. Since T is assumed to be a subspace of W, we know $\alpha L(\mathbf{x}) \in T$. Since L is linear, this means the same as $L(\alpha\mathbf{x}) \in T$. But by def, that means $\alpha\mathbf{x} \in L^{-1}(T)$.

Ch.4.2.16: [Assuming that $L(x) = 0,\ x \neq 0$, prove that A is singular].

OK; this is about the close relationship between L and A, maybe about Ker(L) = N(A), or maybe about nullity. The TFAE theorem may come into it too, because I have to talk about whether A is singular or not, and I don't see any other approach (I don't expect to have any info about det A, for example). So, let's start by re-stating the hypotheses (maybe not really necessary, but this shows the reader that I don't plan anything tricky, like a proof-by-contradiction). Then, I'll try to work A into the proof, and think more about its singular-ness.

Assume that $L(x) = 0$ and $x \neq 0$ and that the nxn A represents L. So, $Ax = 0$.

Good; there's a non-trivial vector in N(A). That rings a bell. But wait - maybe I don't even have to say that. I can go straight to the TFAE theorem.

Since this homogeneous system has a non-trivial solution, the TFAE theorem implies that A is singular. Done.

Just the proof: Assume that $L(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ and that the nxn matrix A represents L. So, $A\mathbf{x} = \mathbf{0}$. Since this homogeneous system has a non-trivial solution, the TFAE theorem implies that A is singular. Done.

Ch 4.3.10: [If A and B are similar, then there are matrices S and T so that A=ST and B=TS].

Hmmm. Another simple algebra exercise using the def of 'similar'? To show that S and T exist, I'll probably have to give a formula for them. I wonder if this S is the same as the S in the def of 'A is similar to B'? Let's try that idea; suppose that $A = S^{-1}BS$. This doesn't look right, because I want A = S times something. The S should be on the left. Probably the other way works better. Let's try that instead.

Assume that B is similar to A, so that $B = S^{-1}AS$. *Aha! Since there is an S on the right side, this fits the $B = TS$ pattern I need.* Set $T = S^{-1}A$ so that $B = TS$. *Now, I have to check that $A = ST$ but that almost HAS to be true, if this exercise is even possible.* Then $ST = S(S^{-1}A) = IA = A$. Done. *Good. That was easier than I expected! I wonder if the converse of this exercise is true ... ? (I won't go into that here, but I'm trying to make the point that you can often learn more from an exercise, even after you solve it. Be curious!).*

Just the proof: Assume that B is similar to A, so that $B = S^{-1}AS$ (for some S). Set $T = S^{-1}A$, so that $B = TS$. Then $ST = S(S^{-1}A) = IA = A$. Done.

Ch.5.1.13: [Decide if $\perp$ is transitive]

Since I don't know the answer yet, I'll look at some examples. First try: $\mathbf{e_1} \perp \mathbf{e_2}$ *and* $\mathbf{e_2} \perp \mathbf{e_3}$*. Is* $\mathbf{e_1} \perp \mathbf{e_3}$ *? Yes. Maybe it is true. But was that a coincidence? I could have chosen lots of other vectors for* $\mathbf{x_3}$*, instead of* $\mathbf{e_3}$*. I don't see why it would HAVE to be* $\perp \mathbf{e_1}$*. What happens if* $\mathbf{x_3} = \mathbf{e_1}$ *? Aha! The conjecture fails! Now, to write the answer clearly:*

The statement can be false. Let $\mathbf{x_1} = \mathbf{e_1}$ and $\mathbf{x_2} = \mathbf{e_2}$ and $\mathbf{x_3} = \mathbf{e_1}$. Then $\mathbf{x_1} \perp \mathbf{x_2}$ and $\mathbf{x_2} \perp \mathbf{x_3}$ but it is false that $\mathbf{x_1} \perp \mathbf{x_3}$.

Sometimes, good examples are hard to find. In fact, the hardest proofs are often ones that require finding an non-obvious example or formula. But in textbook exercises, and casual math-thinking, the odds are usually VERY high that you will find a good example if you just try, perhaps even at random. Maybe you've noticed that yourself from doing MATLAB exercises that use the rand feature.

OK - enough proofs for today. If you learned anything from this page, let me know, and I'll probably write more when I have time.