Notes on some topics from the last week, MAS 3105; updated 4/20/11

This is an outline of some topics from my final Linear Algebra lectures from various years, 2004 to 2011. Some are not in our textbook. Most semesters I don't have time for many of these. You won't be tested on those I didn't cover in class.

1) MATLAB does not compute eigenvalues of A from $p(\lambda)$ like we do. One reason is that it is generally impossible to factor (exactly) a polynomial of degree 5 or higher. Instead:

a) Note that if T is triangular then its eigenvalues are its diagonal entries. Easy!

b) If A is similar to T (see Schur's Theorem) then they have the same eigenvalues, so we'd like to find such a T.

c) We use the GS process to factor $A = QR$, where R is triangular. We set $A_2 = RQ$, which is similar to A, and factor it as Q_2R_2. We repeat indefinitely, setting $A_3 = R_2Q_2 = Q_3R_3$ etc. We get a sequence of similar matrices, and can expect (reasoning omitted) that they become "more and more triangular" and pretty close to T. So, we can take the diagonal entries of one of them (say A_10) to be pretty close to the eigenvalues we want.

2) A few simple exercises for practice:

a) If A has n LI eigenvectors then so do A^T and A^{-1} (if it exists). [the easy proofs are based on the diagonalization of A and facts about similarity from Ch.4.3]

b) If A is 3x3, with only 2 eigenvalues, and each eigenspace has dim = 1, then is A diagonalizable? Ans: No, it only has 2 LI eigenvectors.

3) *The Spectral Decomposition* of A is the formula in exercise 6.4.22 of the 7th Ed. Be able to justify it by multiplying out the formula $A = UDU^H$. Notice that $u_1u_1^H$, for example, is an nxn matrix that projects vectors onto span$\{u_1\}$. So, Ax can be thought of as a linear combination of projections onto eigenvectors.

4) Omitted, but not hard - The very short proof of Cor 6.4.5, and of its converse: If A is real and has a complete orthogonal set of e'vecs [so $A = UDU^T$], then A is symmetric. (try this yourself).

5) Approximation by trig polynomials (see Ch 5.5 and exercises 5.5.28 to 30). the answer was $\pi/2 - 4\cos(x)/\pi$.) Here is a similar problem, from a previous semester.

Find the n^{th} Fourier approximation to $f(t) = t$ on $[0, 2\pi]$.

Solution: The problem asks us to find the projection of the vector $f(t) = t$ onto the span of the vectors (the trig functions) listed on page 280. The formula for the projection $t_n(x)$ is at the top of page 282, and is really the same as the formula in Thm 5.5.7 (the c_i of 5.5.7 are the a_k and b_k of pages 280-282). From integration by parts,

$$b_k = < t, \sin(kt) >= \frac{1}{\pi}\int_0^{2\pi} t\ \sin(kt)\ dt = -k/t$$

and similarly, we get $a_k = 0$ except that $a_0 = \pi/\sqrt{2}$. So the answer is,

$$t_n(x) = \pi + (-2)\sin(t) + (-2/2)\sin(2t) + (-2/3)\sin(3t) + \ldots + (-2/n)\sin(nt)$$

This is a good approximation to f if n is big (and x is not too close to the endpoints of $[0, 2\pi]$). It is like a Taylor polynomial, and is useful in differential equations (etc). The infinite series version is called the Fourier series of $f(t) = t$.

6) A good example to know about is the *nilpotent* matrix below. Check that $A^3 = O$. It has a repeated eigenvalue of 0, and is defective. (see also HW 6.1.7).

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

7) In Spring 2011, we went over the Google application in Ch 6.3 (pg 333).