Notes on some topics from the last week, MAS 3105; updated 4/20/11

This is an outline of some topics from my final Linear Algebra lectures from various years, 2004 to 2011. Some are not in our textbook. Most semesters I don’t have time for many of these. You won’t be tested on those I didn’t cover in class.

1) MATLAB does not compute eigenvalues of \( A \) from \( p(\lambda) \) like we do. One reason is that it is generally impossible to factor (exactly) a polynomial of degree 5 or higher. Instead:

a) Note that if \( T \) is triangular then its eigenvalues are its diagonal entries. Easy!

b) If \( A \) is similar to \( T \) (see Schur’s Theorem) then they have the same eigenvalues, so we’d like to find such a \( T \).

c) We use the GS process to factor \( A = QR \), where \( R \) is triangular. We set \( A_2 = RQ \), which is similar to \( A \), and factor it as \( Q_2R_2 \). We repeat indefinitely, setting \( A_3 = R_2Q_2 = Q_3R_3 \) etc. We get a sequence of similar matrices, and can expect (reasoning omitted) that they become "more and more triangular" and pretty close to \( T \). So, we can take the diagonal entries of one of them (say \( A_{10} \)) to be pretty close to the eigenvalues we want.

2) A few simple exercises for practice:

a) If \( A \) has \( n \) LI eigenvectors then so do \( A^T \) and \( A^{-1} \) (if it exists). [the easy proofs are based on the diagonalization of \( A \) and facts about similarity from Ch.4.3]

b) If \( A \) is 3x3, with only 2 eigenvalues, and each eigenspace has dim = 1, then is \( A \) diagonalizable? Ans: No, it only has 2 LI eigenvectors.

3) The Spectral Decomposition of \( A \) is the formula in exercise 6.4.22 of the 7th Ed. Be able to justify it by multiplying out the formula \( A = UDU^H \). Notice that \( u_1u_1^H \), for example, is an \( n \times n \) matrix that projects vectors onto \( \text{span}\{u_1\} \). So, \( Ax \) can be thought of as a linear combination of projections onto eigenvectors.
4) Omitted, but not hard - The very short proof of Cor 6.4.5, and of its converse: If $A$ is real and has a complete orthogonal set of e'vecs [so $A = UDU^T$], then $A$ is symmetric. (try this yourself).

5) Approximation by trig polynomials (see Ch 5.5 and exercises 5.5.28 to 30). The answer was $\pi/2 - 4 \cos(x)/\pi$.) Here is a similar problem, from a previous semester.

Find the $n^{th}$ Fourier approximation to $f(t) = t$ on $[0, 2\pi]$.

Solution: The problem asks us to find the projection of the vector $f(t) = t$ onto the span of the vectors (the trig functions) listed on page 280. The formula for the projection $t_n(x)$ is at the top of page 282, and is really the same as the formula in Thm 5.5.7 (the $c_i$ of 5.5.7 are the $a_k$ and $b_k$ of pages 280-282). From integration by parts,

$$b_k = \langle t, \sin(kt) \rangle = \frac{1}{\pi} \int_0^{2\pi} t \sin(kt) \, dt = -k/t$$

and similarly, we get $a_k = 0$ except that $a_0 = \pi/\sqrt{2}$. So the answer is,

$$t_n(x) = \pi + (-2) \sin(t) + (2/2) \sin(2t) + (-2/3) \sin(3t) + \ldots + (-2/n) \sin(nt)$$

This is a good approximation to $f$ if $n$ is big (and $x$ is not too close to the endpoints of $[0, 2\pi]$). It is like a Taylor polynomial, and is useful in differential equations (etc). The infinite series version is called the Fourier series of $f(t) = t$.

6) A good example to know about is the nilpotent matrix below. Check that $A^3 = O$. It has a repeated eigenvalue of 0, and is defective. (see also HW 6.1.7).

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

7) In Spring 2011, we went over the Google application in Ch 6.3 (pg 333).