MAA 3200 Sept 22, 2003
Exam I Key Prof. S. Hudson

The average grade was about 70/100, which is normal-to-good. You can use the scale in the syllabus for now. The True-False had the lowest average, at about 60 per cent. The proofs were graded based mainly on correctness and clarity [did you follow a standard strategy ? state your assumptions clearly ? skip steps ? use good notation ?]

1a): If A and $B \backslash C$ are disjoint then $A \cap B \subseteq C$.

Proof: Assume A and $B \backslash C$ are disjoint. ETS $A \cap B \subseteq C$. Let $x \in A \cap B$. ETS $x \in C$. Since $x \in A$ and A is disjoint from $B \backslash C$, $x \notin B \backslash C$. It is false that $x \in B$ and $x \notin C$. But $x \in B$, so $x \notin C$ is false. So, $x \in C$.

I took a slight shortcut with the definition of disjoint here. Notice that I used a sequence of short sentences combined into paragraph form. Try not to let your sentences run on with too many "which means" or "if - then" phrases. Our book uses two columns (givens and goals) to plan out proofs, but this is just "scratch work". Your final proof should not be in column form like that.

Notational mistakes were pretty common, and were sometimes bad enough to make the proof unreadable. Don't confuse predicates, like $x \in A$, with sets, like A. For example,

Good Notation	Bad Notation
$\neg(x \in A \cap B)$	$\neg(A \cap B)$
$A \cap B = \emptyset$	$(x \in A \text{ and } x \in B) = \emptyset$
ETS $A \cap B \subseteq C$	ETS $A \cap B \to C$

1b): If $A \cup B = B$ then $A \cap B = A$.

Proof: Assume $A \cup B = B$. ETS $A \cap B = A$. So, ETS 1) $A \cap B \subseteq A$ and 2) $A \subseteq A \cap B$. Since 1) is a known theorem (easily proved from the definition of $\cap$), we go on to 2). Assume $x \in A$. ETS $x \in A \cap B$. So, ETS $x \in A$ and $x \in B$. Since $x \in A$, we know $x \in A \cup B$. Since $A \cup B = B$, we know $x \in B$. So, $x \in A$ and $x \in B$, and we are done.

2) Answer TRUE or FALSE; you do not have to justify your answers.

$\neg(\exists x \in R, x < 1 \text{ and } x > 2)$
TRUE

$\forall a, b, c \in Z$, if $a|c$ and $b|c$ then $ab|c$.
FALSE (consider a=b=c=5 for example)

$\exists x \in R$, if $x^2 > 1$ then $x^2 < 0$.
TRUE (set x = 0, which makes a $F \to F$, which is true)

$(p \vee q) \rightarrow r$ is logically equivalent to $p \rightarrow r \vee q \rightarrow r$.
FALSE, it is equivalent to $p \rightarrow r \vee q \rightarrow r$ (which is why a proof using cases is valid)

$\exists! x, P(x)$ is equivalent to $\exists x(P(x) \wedge \forall y(P(y) \rightarrow y = x))$.
TRUE (pg 141)

$\forall m \in N, \forall n \in N, \exists p \in N, mn | 2p$.
TRUE - set $p = mn$.

If $mn|3$ then either $m|3$ or $n|3$ ($U = N$).
TRUE - Actually, $m|3$ *and* $n|3$. (I meant to type "If $3|mn \ldots$")

$\forall x \in R[x \neq 2 \rightarrow \exists! y \in R, 2y/(y+1) = x]$.
TRUE (see the theorem on page 146)

$\exists A, B, C$ such that $A \backslash B \subseteq C$ and $\neg(A \subseteq C)$ and $A \cap B = \emptyset$.
FALSE (if $A \cap B = \emptyset$ then $A \backslash B = A$ so $A \subseteq C$, a contradiction).

$\forall A, B, P(A \cup B) = P(A) \cup P(B)$. [$P$ stands for power set.]
FALSE (see problem 7, on page 80, graded in HW2)

3) Two of these are false. Circle them and disprove them by giving counterexamples, and some explanation.

a) $\forall a > 0, \forall b > 0, \exists c > 0, (c < a \wedge c < b)$ (where $U = R$).
True; given a and b, set c = min(a,b)/2.

b) $\forall x \in R, \forall z \in R, \exists y \in R, x^2 + y^2 = z^2$.
False; Let x = 1 and z = 0. There is no y such that $y^2 = -1$.

c) If $A \subseteq B$, $a \in A$, and a and b are not both elements of B then $b \notin B$.
True (see text)

d) If $|x - 3| < 2$ then $|x - 1| < 3$ or $|x - 7| < 3$.
False; Set x=4 and check the inequalities are T F F.

4) Write out the precise definitions of each of these terms or phrases. Use logical notation like "$\forall$" whenever possible.

a) $A \subseteq B$
$\forall x, x \in A \rightarrow x \in B$

b) $\lim_{x \rightarrow a} f(x) = L$
$\forall \epsilon > 0, \exists \delta > 0, \forall x \in R, 0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon$

c) $a|b$
$\exists k \in Z, ak = b$

d) $A \backslash B$
$\{x | x \in A \wedge x \notin B\}$. Or, you could write "$x \in A \backslash B$ means $x \in A$ and $x \notin B$".

Remark on notation - Notice that a), b) and c) are predicates and your definitions should be predicates of the same type. For example, c) is a predicate of the $p(a, b)$ type. If your answer to c) starts with "$\forall a$", it is not the $p(a, b)$ type. Part d) is a *set*, so you should not answer with a *statement*, such as "$\forall x, x \in A \wedge x \notin B$". Try to get the notation right, so people can read your proofs !

5) Let a, b, c be real numbers with $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$. [You can use well-known facts about inequalities and positive numbers. If in doubt about this, see me!]

Proof 1: Assume that $a > b$. We will prove the contrapositive of the statement. Assume that $c > 0$. ETS $ac > bc$. Since $a > b$, we can multiply both sides by c and get $ac > bc$. Done.

Proof 2: Assume that $a > b$ and that $ac \leq bc$. ETS $c \leq 0$. Assume that $c > 0$ to get a contradiction. Since $a > b$, we can multiply both sides by c and get $ac > bc$. This contradicts $ac \leq bc$, so $c > 0$ is false. So, $c \leq 0$.

Remarks: Both of these are indirect proofs, which means they require a little more explanation than a direct proof. I think Proof 1 is better (shorter and clearer) than Proof 2. Proof 2 *must* include the phrase *to get a contradiction* to make the strategy clear.