MAA 3200 — Sept 21, 2010

Exam I and Key — Prof. S. Hudson

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1) [20 pts] Answer True or False. You do not have to justify these.

$\neg(\neg P \to \neg Q) \Leftrightarrow \neg P \wedge Q$

$(\exists k \in Z, 2k = 6) \wedge (\exists k \in Z, 5k = 15)$

If R is a relation on $A = \{1\}$, it must be transitive.

If R is a relation on $A = \{1,\ 2\}$, it must be transitive.

If S, R are relations on a set A, then Dom $(S \circ R) =$ Dom (R)

In the real numbers, $\forall x, \exists y, \forall w, \exists z, x + wyz = 1$.

$\exists! x \in R, (x^2 - 4 = 0 \wedge (x-2)^2 > 2)$

If $n \in Z$ is not prime and $n > 1$, then $2^n - 1$ is not prime.

If $n \in Z$ is not prime and $n > 1$, then $2^n + 1$ is not prime.

If $A \subseteq C$, and B and C are disjoint, and $x \in A$, then $x \notin B$.

2) [15 pts] Show that $A \subseteq B$ iff $P(A) \subseteq P(B)$.

3) [15 pts] Suppose B, C are sets, and for all sets A, $A \cap B = A \cap C$. Show that $B = C$.

4) [20 pts] Prove or disprove ONE of these:
a) If A, B, C, D are sets, and $A \times B \subseteq C \times D$ then $A \subseteq C$.
b) There are infinitely many prime numbers.

5) [15 pts] Show that in Z, if $a|b$ and $a|c$ then $a|(b+c)$.

6) [15 pts] For $x \in R$, prove that if $x^2 \geq x$ then $x \leq 0$ or $x \geq 1$ [use algebra; do not use Calculus].

Bonus [approx 5 pts]: Prove that if $\lim_{x\to 3} f(x) = 2$ then $\exists c > 0,\ |x-3| < c \to f(x) > 1$. Include the definition of limit in your proof.

Remarks and Answers The average score [based on the top 20 grades] was approx 61/100, which is a bit lower than expected. The worst results were on Problem 3 (33% correct) and on Problem 4 (53% correct), and the best were on Problem 5 (92% correct). The [approximate] scale for the exam is:

A's 76 to 100
B's 63 to 75
C's 53 to 62

D's 43 to 52

This scale is approximate because it depends on the average score above, which may change slightly if people drop. It is unofficial in the sense that the only scale that really matters is the one at the end. But having said that, I set all my scales in about the same way, so this one is fairly dependable.

This exam tested you mainly on basic proof-writing skills. A) Problems 3 and 6 tested your ability to handle $\forall$ and $\vee$ sentences [eg less-direct proofs]. B) Problems 2 and 5 were fairly simple direct proofs, but tested you on some recently-learned notation (power sets and $a|b$). C) Problem 4 gave you a choice of two slightly harder proofs from the lectures/reading, and tested how much you learned from those sources. If you go on in advanced mathematics, you will need these three kinds of skills. Hopefully, you are learning the basic strategies (A) this semester. There is no end to new notation (B), but you will probably learn to digest it faster with more practice [as you become more comfortable with the common logical phrases such as $\exists$ etc]. Eventually, the focus will be on the main theorems and techniques of the subject (more like C).

1) TTTFF FTTFT (see me for explanations)

2) This should be a happy mix of words and formulas. Some people tried to do it without words, and others without enough formulas. It does not require any unusual strategies, such as proof-by-contradiction [if you do it that way, say so! and explain extra carefully]. Most people handled iff correctly, and split their proof into two parts, but often got lost with the power sets, the notation or the next strategy (using 'ETS' often is a good way to stay on track). Here's the first half, with non-essential comments in brackets.

Proof of $\Rightarrow$: Assume $A \subseteq B$. ETS $P(A) \subseteq P(B)$. [this will be a direct proof; I will focus on the conclusion for a while, because it is more complex]. Let $X \in P(A)$, which means $X \subseteq A$. ETS $X \in P(B)$ [am using the def of $\subseteq$ in the previous ETS, to convert to an 'implies' statement, and another direct proof strategy.] This means ETS $X \subseteq B$. But we know $X \subseteq A$ and $A \subseteq B$, so $X \subseteq B$, by the transitive property [in my opinion, using this property is OK here, but if not, we easily prove $X \subseteq B$ by assuming $x \in X$, etc]. Done. I am leaving the second half to you.

3) This mainly tests if you know how to **use** a statement of the form $\forall A, p(A)$. Usually, you substitute in some specific value (in this case, a set) for A, and you get to use $p(A)$. You may do this more than once.

Proof 1: Let B, C be arbitrary sets. Assume $\forall A, A \cap B = A \cap C$. Set $A = U$, the universal set [If you set $A = B \cup C$, the rest will be the same. If neither of these choices occurred to you, see Proof 2 below.] From our assumption, we get $U \cap B = U \cap C$ which means $B = C$. Done. [Notice that I didn't have to prove $B \subseteq C$ and $C \subseteq B$ separately this time; it is always OK to do that, but in this example, it just takes longer].

Proof 2: Let B, C be arbitrary sets. Assume $\forall A, A \cap B = A \cap C$. Set $A = B$ [this choice may involve some trial and error, which you may do on scratch paper, and there may be several good choices]. From our assumption, we get $B = B \cap C$. Now set $A = C$ and get $B \cap C = C$. This shows $B = C$. Done.

If you did not set $A =$ something, I didn't give you much partial credit. I don't think there is any way to prove this example without that basic method. Also, see the end of this answer key.

4) These were done in class (and in the text). Part a) is false, because B might be empty (but you should specify the other sets too, to form a clear counterexample). A common mistake was to "prove" part a) using the faulty proof discussed in class.

5) Assume $a|b$ and $a|c$. So, $\exists k, l \in Z, ak = b, al = c$. So, $b + c = ak + al = a(k + l)$. Since $k + l \in Z$ this shows $a|(b + c)$. Most people got this one right. The most common minor problems were with wording (eg not enough words, or treating $a|(b+c)$ like an assumption, by discussing what it implies).

Also, $ak = b$ should not be written as $a = b/k$, since the universal set here is Z. The b/k notation is used in other universal sets, such as Q and R. This a fairly minor issue, and it is sometimes OK to change the universal set in the middle of a proof with proper explanation, so I didn't deduct any points for this (but be more careful with this in the future). Another complaint about $a = b/k$: if you use this notation, explain why $k \neq 0$.

6) *Proof 1:* Assume $x^2 \geq x$ and $x > 0$. ETS $x \geq 1$. [This is one of our standard methods to prove $p \vee q$ - we prove $\neg p \rightarrow q$ instead]. Since $x > 0$, we can divide $x^2 \geq x$ by x to get $x \geq 1$. Done.

Proof 2: [Contrapositive]. Assume $x < 1$ and $x > 0$. ETS $x^2 < x$. Since $x > 0$, we can multiply $x < 1$ by x to get $x^2 < x$. Done. [Yes, these two proofs are very similar].

A common mistake was to confuse p and q (for example, to prove $x^2 \geq x$). Another was to use cases incorrectly. It is never "wrong" to split a proof into cases, but I don't see a good reason to do so here [and most attempts that I graded did not work out well].

Bonus) Rough idea ; we want $|f(x) - 2| < \epsilon$ [from the def of limit] to imply $f(x) > 1$. This seems reasonable since the first inequality says $x \approx 2$ [roughly]. It seems that if $\epsilon \leq 1$ this should work out. But we need to go through the logic, to see if we get to choose ϵ, c, etc.

Proof: Assume that $\lim_{x \to 3} f(x) = 2$, which means $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x, 0 < |x - 3| < \delta \rightarrow |f(x) - 2| < \epsilon$. ETS $\exists c > 0$, $|x - 3| < c \rightarrow f(x) > 1$. Set $\epsilon = 1$. [We are given a $\forall \epsilon$ statement, so we can choose it as we like (see problem 3 for more on this). See the rough proof above about why I'm choosing it be 1. None of this has to be explained to the reader yet, but the choice has to be justified by what follows]. Fix some $\delta > 0$ that works for this choice [so, $\forall x, 0 < |x - 3| < \delta \rightarrow |f(x) - 2| < 1$] Set $c = \delta$ [I am finally looking at the ETS. Couldn't do it any earlier, because I had no good formula for c until now. I hope you already see why I chose $c = \delta$. If not, be patient!]. Assume $|x - 3| < c$ [following the structure of the ETS, and planning a direct proof; the new ETS is $f(x) > 1$]. From assumptions and choice of $c = \delta$, we get $|f(x) - 2| < 1$, which implies $-1 < f(x) - 2 < 1$. Adding 2, this implies $f(x) > 1$. Done.

Remark: When writing this proof, I noticed a small flaw in the problem [and in my proof]. Did you spot it? I will give a little extra credit to the first student who finds it by themselves. This is one of the many benefits of careful proofs; if a statement has a tiny error, the proof effort should help us spot it. Second remark: I don't usually give much partial credit on problems worth only 5 points, such as extra credit, though I did give some on this one.

Footnote to problem 3): You have probably used $\forall$ sentences many times. In fact most theorems are of this type. Here's a very easy example. Given that $\sin(\theta) = \sqrt{8}/3$, prove that $|\cos(\theta)| = 1/3$.

Proof: We know the theorem, $\forall x, \sin^2(x) + \cos^2(x) = 1$. Assume $\sin(\theta) = \sqrt{8}/3$ and set $x = \theta$. We get $8/9 + \cos^2(\theta) = 1$, and the rest is easy.

Notice the similarity between this proof and the proof of problem 3). This semester, you need to examine many simple proofs like these and look for such patterns. There really aren't very many, and most are covered in Chapter 3. That chapter emphasizes how to *prove* various sentence types. Also, pay attention to how the different types are *used*; this may have been the issue with 3).