MAA 3200 Sept 24, 2013
Exam I and Key Prof. S. Hudson

1) [20 pts] Answer True or False. You do not have to justify these.

$\neg(P \to \neg Q) \Leftrightarrow P \wedge Q$

$(\exists k \in Z, 2k = 8) \wedge (\exists k \in Z, 5k = 25)$

$\exists k \in Z, (2k = 8 \wedge 5k = 25)$

$((P \vee Q) \wedge \neg P) \wedge \neg Q$ is a contradiction.

$\exists! x \in R, (x-1)(x-3)(x-5) = 0 \wedge |x-2| < 2$

If $n \geq 7$ is an integer, then there are positive integers k, j so that $n = 2k + 3j$.

There is a set that is a member of every set.

In the real numbers, $\forall x, \exists y, \forall w, wxy < w^2$.

$\emptyset \subseteq \{\emptyset\}$

If A is a proper subset of $\emptyset$ then $A = \{17\}$.

2) [15 pts] Let A, B be sets with $P(A \cup B) = P(A) \cup P(B)$ (as usual, P refers to the power set). Prove that $A \subseteq B$ or $B \subseteq A$.

3) [15 pts] Prove that for all integers $n \geq 1$,

$$2 + 5 + 8 + \cdots + (3n - 1) = n(3n + 1)/2$$

4) [10 pts] Let $A_n = [5 - \frac{1}{n}, 5 + \frac{1}{n}] \cup (7 - \frac{1}{n}, 7 + \frac{1}{n})$. Find $\cap_{n \in N} A_n$.

The next few problems are intended to be fairly easy and short. First answer with *True* or *False* and then justify, perhaps by giving a counterexample.

5) [10 pts] Prove or disprove: If n is prime then $2^n + 1$ is prime.

6) [10 pts] Prove or disprove: If n is odd and m is even, then $n + m$ is odd.

7) [10 pts] Prove or disprove: For all sets A, B, C, if $A - B \subseteq C - B$ then $A \subseteq C$.

8) [10 pts] Prove or disprove (for real numbers): $\forall \epsilon > 0, \exists \delta > 0, \delta + \epsilon < 2\epsilon^2$.

Bonus [approx 5 pts]: Prove that $\lim_{x \to 3} 4x - 5 = 10$ is false, using the definition of limit.

Remarks and Answers: Problems 5 and 6 turned out extremely well for almost everyone, while problems 2 and 7 (and to a lesser extent 8) turned out very bad. So, the difference between an A and an F depends mainly on the other 4 problems. The average was 61 (a bit low), based on the top 10 scores, with a high score of 75. An approximate unofficial scale is:

A's 72 - 100
B's 62 - 71
C's 52 - 61
D's 42 - 51

1) TTFTF TFFTT

2) Here is the proof from class:

Assume $P(A \cup B) = P(A) \cup P(B)$ and $A \not\subseteq B$ and $B \not\subseteq A$, to get a contradiction. So, $\exists x \in A - B$ and $\exists y \in B - A$. Then $\{x, y\} \in P(A \cup B)$ but $\{x, y\} \notin P(A)$ and $\{x, y\} \notin P(B)$. This contradicts $P(A \cup B) = P(A) \cup P(B)$. Done.

Most answers did not resemble this one, did not follow any clear strategy and did not get much partial credit. The results were so low that, because of scaling, a score of 0 won't really hurt your grade (my interpretation is that the problem was too hard, for whatever reason). The one student who got it right gets, in effect, 15 points extra credit.

3) See similar examples from Ch 3. The most common problem was a lack of explanation. Especially, distinguish assumptions from goals. I don't know any hard and fast rules on what HAS to be in a proof, but in addition to formulas, your answer should include most of these phrases, or similar ones:

We will use induction on n.
The Basis step is the case $n = 1$.
Induction Step: Let $n \geq 1$ be an integer.
Assume [insert the given formula] is true.
ETS [insert the given formula but with $n + 1$ subbed for n] is true.

Most of the remaining formulas should be prefaced by 'So'.

4) $\{5, 7\}$ (since 5 belongs to every interval $[5 - \frac{1}{n}, 5 + \frac{1}{n}]$, etc).

I intended this to be a little more interesting, maybe with $A_n = [5 - \frac{1}{n}, 5 + \frac{1}{n}] \cup (7, 7 + \frac{1}{n})$ instead. Then, the intersection is just $\{5\}$.

5) False. $n = 3$ is prime but $2^3 + 1 = 9$ is not. The results were good on 5) and on 6).

6) True. By the definitions of odd and even, we can let $n = 2k + 1$ and $m = 2j$. Then $n + m = 2(k + j) + 1$, which is odd, since $j + k$ is an integer.

Most of the proofs were good; one rather common mistake was to over-use a letter. For example, $m = 2k$ would be incorrect.

7) False. Let $A = B = \{5\}$ and let $C = \emptyset$. Then $A - B = C = C - B$. But $A \not\subseteq C$. There are many similar examples.

This was not supposed to be tricky, but *everyone* thought it was true! In this course you will learn to be more skeptical (Bertrand Russell, among others, showed us how) and to value a proof as THE sure-fire way to resolve doubts!

Like problem 2 (and problems 5-6 but for the opposite reason), this one will not actually affect your grade much.

8) False. A counterexample is $\epsilon = 0.5$ (then $\delta + \epsilon < 2\epsilon^2$ leads to $\delta < 0$, a contradiction).

B) See the similar example from my lecture. To get started, observe that $|f(3) - 10| = 3$ so you can set $\epsilon = 1$ (anything less than 3 should be OK). Then show no δ works out.