MAA 3200 Sept 27, 2018
Exam I Prof. S. Hudson

Regard A, B as arbitrary sets and R as an arbitrary relation on A, etc, unless stated otherwise. Let $\mathcal{R}$ be the real numbers, let N be the natural numbers and let Z be the integers.

1) [20 pts] Answer True or False. You do not have to justify these.

$A \setminus B$ and $B \setminus A$ are disjoint.

If $f : A \to B$ and $f^{-1} : B \to A$ then f^{-1} is onto (surjective).

$\forall m \in Z, \exists k \in Z, m + 2k > k + 10$.

$(P \wedge Q) \to \neg P$ is a contradiction.

dom $R =$ ran R^{-1}.

If R is transitive then $R \circ R \subseteq R$.

The relation '$<$' is a total order on the real numbers, $\mathcal{R}$.

The relation $R = \{(x, y) \in N \times N | x^2 \leq y \leq x\}$ on N is a function, $R : N \to N$.

Suppose $(a, b)R(c, d)$ means $a + d = b + c$, for $a, b, c, d \in N$. Then R is an equivalence relation on $N \times N$.

One valid strategy to prove $p \to q$ is to assume $\neg p$ and then prove $\neg q$.

2) [15 pts] Let A, B and C be sets such that $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Prove that $A \subseteq B$.

3) [15 pts] Let R be a reflexive relation on A. Prove that $R \subseteq R \circ R$.

4) [10 pts] Define $f : \mathcal{R} \to \mathcal{P}(\mathcal{R})$ (the power set of the real numbers) by the formula $f(x) = \{y \in \mathcal{R} |\ y^2 < x\}$.

4a) Find $f(2)$ and simplify slightly.

4b) Is f 1-1 ? Explain briefly.

4c) Is f onto ? Explain briefly.

5) [10 pts] Answer with *True* or *False* and then prove or disprove: If R is an anti-symmetric relation on $A = \{a, b, c\}$ then R cannot also be symmetric.

6) [10 pts] Choose ONE to prove using induction (or strong induction). Circle it and do it.

6a) State and prove the Well-Ordering Principle for the natural numbers.

6b) Prove that for $n \geq 5$, $2^n \geq n^2$. If you cannot get the algebra exactly right, explain clearly what you want to do, for partial credit.

7) [10 pts] Prove that there are infinitely many prime numbers, as done in class. Two small hints; 1) this was an indirect proof, and 2) the formula $m = p_1 p_2 \cdots p_k + 1$ was useful.

8) [10 pts] Define f on the real numbers by $f(x) = x/2$. Let $A_0 = [2, 4]$. Define A_n recursively for $n = 1, 2, \ldots$ by $A_n = f(A_{n-1})$. Find A_3.

Bonus [5 pts]: How many equivalence relations are there on the set $A = \{a, b, c, d\}$?

Remarks and Answers: The average among the top 19 scores was approx 71%, with high scores of 99 and 97. The best results were on problems 1 and 8 (approx 90% or more). The worst results were on problems 3, 4, and 7 (approx 50% to 55%). Here is an advisory scale for the exam:

A's 78 to 100
B's 68 to 77
C's 58 to 67
D's 48 to 57

1) TTTFT TFFTF. The results were very good on this, better than normal even for True-False.

2) Proof: Assume A, B and C be sets such that $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. ETS $A \subseteq B$. Let $x \in A$. ETS $x \in B$. (see Remark 1 below).

Case I: Suppose $x \in C$. We assumed $x \in A$ earlier, so $x \in A \cap C \subseteq B \cap C \subseteq B$. So, $x \in B$. [Ending Case I, because we reached our ETS. I would not say much more yet, such as "So, $A \subseteq B$". That is not fully proven until after Case II is complete.]

Case II: Suppose $x \notin C$. We know $x \in A \subseteq A \cup C \subseteq B \cup C$, so $x \in B$ or $x \in C$. Since $x \notin C$ we know $x \in B$. So, in either case, $x \in B$. Done.

Remark 1: I do not feel it is necessary to mention the definition of $A \subseteq B$ since we have used it so often already, but of course it is OK to do that. The 2 ETS's are somewhat optional, but many answers had no clear "direction" and ETS's should help with that. At this point it is hard to make progress not knowing whether $x \in C$ or not, so it makes sense to use cases. If you like, you can insert "Either $x \in C$ or $x \notin C$" here, for just a little more clarity, but just writing "Case I:" and "Case II:" is pretty standard].

Other remarks: There were some good answers, but not many with both correct logic and good style (a sequence of sentences, including formulas like $x \in A$). Some people deduced quickly that $x \in B$ or $x \in C$. In effect, Case II then becomes "$x \in B$. Done", so maybe this gives a slightly simpler proof. A few people used cases incorrectly. You can use cases when given a $p \vee q$ hypothesis, but not with $p \wedge q$. So, "Case I: $A \cap C \subseteq B \cap C$" is a mistake. I suggest that beginners normally write "Case I: Suppose [. . . some p] and later write "Case II: Suppose [... $\neg p$]". There are other ways to use cases, but if you are struggling with the concept, this might help.

It is not OK in this proof to set $C = U$ (or any other set). That is a bit like giving an example instead of a proof. It is OK to choose C in certain examples, where you are given $\forall C \cdots$ as a hypothesis (also in examples with ETS $\exists C$). A proof by contradiction is always possible, but the logic and the explanations are typically harder, and such answers did not usually turn out well.

3) Proof: Let R be a reflexive relation on A. ETS: $R \subseteq R \circ R$. Let $(a, b) \in R$. ETS: $(a, b) \in R \circ R$. Since R is reflexive, $(b, b) \in R$. Since $(a, b) \in R$ and $(b, b) \in R$, we know $(a, b) \in R \circ R$ (by the definition of $R \circ R$). Done.

Very few people wrote "Let $(a, b) \in R$". It is OK to write "Let $x \in R$" and then infer that $\exists a, b \in A, x = (a, b)$. It is not OK to infer that $a = b$. That is not the definition of reflexive (for example $\leq$ is reflexive and $3 \leq 4$ and $3 \neq 4$). For the same reason, it is not OK to write "Let $(a, a) \in R$".

Note: The sentence "Let $(a, a) \in R$" is not automatically wrong. It might be OK with some totally different proof strategy (but I can't imagine one). So, your grader might not mark "Let $(a, a) \in R$" with a big X. Instead, you might find a question mark and/or a note at the end of your answer, like "This is not a proof" or "I can't follow this". If you get a low score on a proof without any step marked with an X, it is probably due to a faulty (or unclear) proof strategy.

4) Parts a, b, c were worth 4, 3, 3 points. In b and c, you had to say "No" to get any partial credit. Then, most of the credit was for the explanation.
4a) $(-\sqrt{2}, \sqrt{2})$.
4b) No. $f(-1) = \emptyset$ and $f(-2) = \emptyset$. Few people made the connection from $y^2 < -1$ to $\emptyset$. If we restrict the domain of this function to $x > 0$ then we do get a 1-1 function.
4c) No. For example, there is no x such that $f(x) = \mathcal{R}$. A specific counterexample like this is clearer than a general discussion.

5) False. For example, $R = \{(a, a), (b, b), (c, c)\}$ is both anti-symmetric and symmetric (and $R = \emptyset$ may be an even simpler counterexample).

A remark on the logic, a reminder: the proposition takes the form $\forall R, p(R)$. The negation takes the form $\exists R, \neg p(R)$. The standard method to prove this negation is to give an example of R that makes the original claim false (eg a counterexample). It is hard to imagine another method for this example. We will see some alternatives methods for proving '$\exists x$' later this term, but giving an example is the most common.

6) See the text for the outline and the wording. The algebra part of the induction step goes like this; $2^{n+1} = 2 \cdot 2^n \geq 2n^2 \geq n^2 + 5n \geq n^2 + 2n + 1 = (n + 1)^2$. Most of these steps should be justified briefly.

7) See the lecture notes or the textbook.

8) $[1/4, 1/2]$

Bonus) 15, because there are 15 ways to partition A. You could list them all, but to save time you can group some together. For example, there are 3 partitions that split A into two subsets with two elements each, such as $\{a, d\} \cup \{b, c\}$. Etc. Notice that $\{c, b\} \cup \{a, d\}$ is the same and should not be counted separately.