MAA 3200 October 20, 2003
Exam II Key Prof. S. Hudson

1) [15 pts] Choose ONE to prove:

a) Every integer $n > 1$ is a prime or a product of primes.

b) Prove that $A \sim B$ (the equinumerable relation) is an equivalence relation.

2) [15 pts] Let $A = \{a, b, c, d\}$.
a) How many different relations $R$ are possible on $A$? [Hint: recall that $|P(S)| = 2^{|S|}$.]

b) How many different equivalence relations are possible?

3)[15 pts] Let $R$ and $S$ be relations on a set $A$. Prove or disprove: $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

4)[15 pts] Prove or disprove: If $f : A \to B$ and $S, T \subseteq A$ then $f(S \cup T) = f(S) \cup f(T)$.

5) [15 pts] Prove that for all $n \in N$, $3|n^3 - n$. Use induction.

6) [25pts] Answer True or False: You don't have to explain.
If $f : A \to B$ then $f^{-1}$ is a relation and its domain is $B$.
If $A \times B \sim C \times D$ and $A \sim C$ then $B \sim D$.
If $A$ and $B$ are denumerable then so is $A \cup B$.
If $f$ and $g$ are functions from $R$ to itself and $f \circ g$ is injective (1-1), then $g$ is injective.
If $R$ and $S$ are equivalence relations on $A$, and $A/R = A/S$, then $R = S$.

BONUS (5pts): Assume $f : A \to B$ and $\forall$ sets $M \subseteq A$, $M = f^{-1}(f(M))$. Prove $f$ is injective.

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**Answers:** The class average was about 61/100, with the lowest average score on problem 2 (which is relatively unimportant) and high grades on the induction proof. The cleanest proofs of 3) and 4) use *cases*. One common problem was excessive use of bad notation, which led to serious mistakes, or to unreadable proofs.

1) Both of these are in the text.

2a) A relation is an $R \subseteq A \times A$, so $R$ is an element of $P(A \times A)$. So, there are $2^{|A \times A|} = 2^{16}$ possible $R$'s.

2b) There are 15. This is similar to HW 1-2, page 213. You can list all 15 partitions, or group the partitions according to the size of the equivalence classes, as below. It is easy to make small mistakes here, so I mainly graded your method.

There are 4 different 3+1 splits (eg $\{a, b, c\}$ and $\{d\}$).
There are 3 different 2+2 splits (eg $\{b, c\}$ and $\{a, d\}$).
There are 6 different 2+1+1 splits.
There are 1 different 1+1+1+1 splits.
There are 1 different 4+0 splits. The total is 15.

3) True. Use the "containment both ways" strategy for proving two sets are equal. In problems 3 and 4, some people wrote phrases like "Assume $(R \cup S)^{-1}$". This makes no sense - you can't assume a set!

Part 1) ($\subseteq$): Let $(a, b) \in (R \cup S)^{-1}$. So, $(b, a) \in R \cup S$. So, $(b, a) \in R$ or $(b, a) \in S$.
Case 1: If $(b, a) \in R$ then $(a, b) \in R^{-1} \subseteq R^{-1} \cup S^{-1}$.
Case 2: If $(b, a) \in S$ then $(a, b) \in S^{-1} \subseteq R^{-1} \cup S^{-1}$. Done with Part 1).
Part 2) is similar, and left to you. It also uses cases.

4) True. The proof is a lot like that of 3). Use the "containment both ways" strategy for proving two sets are equal (and use cases). Again, do NOT write things like "ETS $f(S) \cup f(T)$" which makes no sense.

Part 1) ($\subseteq$): Let $y \in f(S \cup T)$ which means $\exists x \in S \cup T : f(x) = y$.
If $x \in S$ then $y \in f(S) \subseteq f(S) \cup f(T)$.
Likewise, if $x \in T$ then $y \in f(T) \subseteq f(S) \cup f(T)$. Done with part 1).

Part 2) is left to you.

5) This was done in class, and on page 248. Good results here !

6) FFTTT Part a) would be true if $f$ were *onto*. Parts b) and e) were HW. Part c) is in thm 7.2.1.

Bonus: Let $f(a) = f(b)$. ETS $a = b$. Let $M = \{a\}$. So, $f(b) = f(a) \in f(M)$. So, $b \in f^{-1}(f(M)) = M = \{a\}$. So, $b = a$.