MAA 3200 — Oct 19, 2010

Exam II and key — Prof. S. Hudson

1) [25 points; the others problems are 15 each] Give examples [or explain why none exists];

a) A set B, and a 1-1 function $f : B \to B$, which is not onto.

b) A non-empty relation R on a set A which is symmetric and anti-symmetric, but not reflexive.

c) Countable sets A and B which are not equinumerable.

d) A set B with cardinality greater than R (so, $R \prec B$).

e) A subset of $Q \times Q$ which is not countable.

2) Suppose R is a transitive relation on A. Show that R^{-1} is also transitive on A.

3) Suppose $f : R \to R$ (the real numbers) is defined by $f(x) = mx + b$ where $m \neq 0$. Show that f is a 1-1 correspondence.

4) Explain what is wrong with this proof by induction, that all rabbits are the same color:

Proof: We will let $p(n)$ be the statement that any given set of n rabbits are all the same color. If we can prove this for all $n \geq 1$ then we are done (since there only finitely many rabbits in the world). Clearly, if we have only one rabbit, it is the same color as itself, and $p(1)$ is true. Now, suppose that $p(n)$ is true, and we will prove $p(n+1)$. So, suppose we have a set of $n+1$ rabbits, $A = \{r_1, r_2, \ldots r_{n+1}\}$. By the induction hypothesis, we know the rabbits $\{r_1, r_2, \ldots r_n\}$ are all the same color. Likewise, the rabbits $\{r_2, \ldots r_{n+1}\}$ are all the same color. So, all the rabbits in A also have the same color.

You can ignore any minor error that can be easily fixed, but must only find the most serious one. If you cannot explain the error, underline the first step that seems seriously wrong (for partial credit).

5) Choose ONE proof;

a) $R \setminus Q \sim R$. For full credit don't use any tools (theorems, etc) beyond Ch 7.2.

b) $(0, 1)$ is uncountable, using Cantor's Diagonal Argument.

c) Any subset of any I_n is finite. [HW 7.1.8a]. Remark: Use only tools from Ch.7.1 or earlier (I'd use induction on n, but maybe there is some other way).

6) Choose ONE textbook proof;

a) Thm 5.3.1; if $f : A \to B$ is one-to-one and onto, then $f^{-1} : B \to A$.

b) State the Well-Ordering Principle and prove it using induction or strong induction. [Hint: the WOP is partly about whether $S = \emptyset$, which is equivalent to $\forall n \in N,\ n \notin S$].

Bonus) [about 5 pts]; Give a specific example of a 1-1 corr $f : (0, 1) \to [0, 1]$.

Key: The average among the top 15 was approx 53/100, which is pretty low. I think part of that was Problem 1; it is often hard to think of examples on the spot (though I believe examples are important in learning a subject, and suggest including this in your study). The worst results were on Problems 4 and 5 and the best were on 6. For now, the scale is:

A's 70 to 100
B's 55 to 69
C's 45 to 54
D's 35 to 44

I may lower the scale a bit if I can determine that most students are working hard enough on this course. So far, I have had few visitors to office hours, not many questions by email, and not many students have been visiting our LA. Roughly half of the students taking the exam did not hand in HW3. If some of these indicators improve, I will lower the scale.

1a) This is impossible if B is finite, but fairly easy if you try an infinite set, such as $B = N$. Set $f(n) = n + 1$ or $= n^2$, etc.

1b) Let $A = \{1, 2\}$ and $R = \{(1, 1)\}$. Check that this works!

1c) Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$.

1d) Let $B = P(R)$ (power set of the reals).

1e) None exists, because $Q \times Q$ is countable.

2) **Proof:** Assume R is transitive. ETS R^{-1} is transitive, which means $\forall a, b, c \in A$, if $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$ then $(a, c) \in R^{-1}$. Let $a, b, c \in A$ be arbitrary. Assume $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. ETS $(a, c) \in R^{-1}$. Since, $(a, b) \in R^{-1}$, we know $(b, a) \in R$ by definition of R^{-1}. Likewise, $(c, b) \in R$. By transitivity, $(c, a) \in R$. So, $(a, c) \in R^{-1}$, by definition of R^{-1} This was our goal, so we are done.

Since this used only direct methods, I did not mention any proof strategy by name. I would accept a slightly less formal proof that used the same logic. For example:

Casual Proof: We must show that if $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$ then $(a, c) \in R^{-1}$. Assume $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$ so that $(b, a) \in R^{-1}$ and $(c, b) \in R$. By transitivity, $(c, a) \in R$, so $(a, c) \in R^{-1}$ and done.

This is pretty minimal, but would make sense to any 'pro', who could easily fill in the minor gaps. You should probably avoid this casual style until you have mastered the basic methods, and can easily keep track of the ETS in your head. For now, write down all your ETS's and $\forall$'s, etc. Most people in this exam were careless and wrote something like this:

Poor Proof: Assume R is transitive. So, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. So, $(b, a) \in R^{-1}$ and $(c, b) \in R^{-1}$ and $(c, a) \in R^{-1}$ by definition of R^{-1}. Since a, b, c were arbitrary, this shows R^{-1} is transitive.

A careful study of the poor proof shows that the assumptions were: $(a, b) \in R$ and $(b, c) \in R$. The conclusions were: $(a, c) \in R$ and $(b, a) \in R^{-1}$ and $(c, b) \in R^{-1}$ and $(c, a) \in R^{-1}$. In other words, the poor proof shows:

$$\forall a, b, c \in A,\ (a, b) \in R \wedge (b, c) \in R \rightarrow (a, c) \in R \wedge (b, a) \in R^{-1} \wedge (c, b) \in R^{-1} \wedge (c, a) \in R^{-1}$$

The proof *should* show something equivalent to the definition of 'R^{-1} is transitive', such as:

$$\forall a, b, c \in A,\ (b, a) \in R^{-1} \wedge (c, b) \in R^{-1} \rightarrow (c, a) \in R^{-1}$$

Since the poor proof is not TOO far off, I gave about 9-10 points partial credit, out of 15. At least it uses correct definitions of transitive and R^{-1}, and it is not too hard to see what the writer is thinking. In most poor proofs, it is hard to say exactly what's wrong; it simply lacks a clear plan. I repeat some suggestions:

i) Write down your ETS clearly, and if it changes during the proof, write down the new one clearly.

ii) Usually, you should focus more on your ETS than on your assumption(s), especially when p and/or q are $\forall$ statements. In this case 'transitive' is a $\forall$ statement that appears twice (as an assumption and as a goal). Normally, you have to *wait* to *use* a $\forall$ (eventually you plug into it). In this case, you wait to use the assumption, that R is transitive.

iii) Memorize and internalize Velleman's summary, pages 376-379.

3) This problem is not hard and most of the proofs were OK. But recall that *one-to-one correspondence* means *one-to-one AND onto.* Many people did not prove onto. Get this kind of error out of your system ! Vocabulary is very important in advanced mathematics, and we have been using the phrase *one-to-one correspondence* since Chapter 5, and especially often in Ch 7. Your professors will probably not remind you to learn such definitions; this is a basic minimal requirement in advanced courses.

4) The problem is with the last sentence. The idea is that since r_2 is in both sets, all the rabbits have the same color as r_2. This idea is OK when $n > 1$, but when $n = 1$, the first set is just $\{r_1\}$, which doesn't contain r_2. So, the idea fails.

I gave partial credit for questioning whether the sets overlap, but full credit only if you pointed to the real problem case, $n = 1$.

This error was hard to find, partly because the proof was written such a casual style. If it included more explanations, like 'Let $n \geq 1$' and 'since r_2 belongs to both sets', we'd be more likely to question that last step, and spot the error. This kind of proof teaches us to be more rigorous.

On the other hand, it is clear from the start that something is wrong, since not all rabbits are the same color. One way to find the mistake is to ask yourself which $p(n)$'s are true. Clearly, $p(1)$ is true but $p(2)$ is not. So, the inductive step $p(n) \rightarrow p(n + 1)$ is false when $n = 1$. So, we examine that part of the proof with $n = 1$ in mind, and can hopefully find the error quickly.

5) Part b) was in the lectures (and more-or-less in the book). Parts a) and c) are a bit difficult, but were in the assigned HW, so I'd hoped people were ready for at least one of them. Here's a rough proof of a) which is HW 7.2.1b. The idea is that the 2 sets differ by a countable set Q, and we have seen some examples where a countable set doesn't matter too much (such as $N \sim Z$).

Rough Proof: Let $S = \{x \in R : x + \sqrt{2} \in Q\}$. I leave it as exercises for you that $S \sim Q$ (using $f(x) = x - \sqrt{2}$) and that $S \cap Q = \emptyset$. Also, that $S \cup Q \sim S$ (use thm 7.2.2). Let $A = R \setminus (S \cup Q)$. Apply Thm 7.1.2(2) to get $A \cup S \cup Q \sim A \cup S$, which is $R \sim R \setminus Q$.

One moral here is to make sure you understand each HW problem, ideally before the HW due date, but definitely before the exam. As always, you are welcome to ask me or Alejandro for help with this.

6) See the textbook.

B) This is similar to problem 5a. See me if interested in more detail.