MAA 3200 Oct 25, 2018
Exam II Prof. S. Hudson

1) [5 pts each] Short answer:

Define the relation on sets, $A \preceq B$ (from Velleman, Ch 7.3). Also define the relation $A \prec B$.

Define the set of rational numbers as done in class. You can use N and/or Z for this. You do not have to define any operations or partial orders, just the set. But include any needed equivalence relation, etc.

Discuss briefly what it means for a number system to be *unique*, for example, when we say that R is the unique complete ordered field. (what if it were constructed a different way or in another language, etc?)

Discuss briefly what it means for an operation on equivalence classes (for example, + on Z) to be *well-defined*. Using an example is OK.

2) [5 pts each] Short answer:

State the Completeness Axiom of the real numbers.

How was the sequence of functions $p_m(n)$ used in our NZQ work ? (ex: to define something? or just in one specific proof ?) Compute $p_4(5)$.

3) [20 pts] Answer True or False. You do not have to justify these.

Every nonempty bounded set in Q has a least upper bound.

Every set $S \subseteq (-\infty, 19]$ has a least upper bound in R.

The set of all finite sequences of rational numbers (such as $2/3, -1/2, 17$) is countable.

In a field, every nonzero element x has a multiplicative inverse x^{-1}.

The trichotomy principle holds in N, Z, Q, and R.

Z_4 is a field.

Every Cauchy sequence in an ordered field F converges in that field.

The sequence $a_n = \frac{5n+7(-1)^n}{6n+8(-1)^n}$ converges.

The sequence $a_n = \ln(n+1)$ is Cauchy.

The power set of $[0, 1]$ is countable.

4) Suppose I and J are intervals. Under what conditions must $I \cup J$ also be an interval ? Give necessary and sufficient conditions, if possible.

5) Prove that $\forall \epsilon > 0, \exists \delta > 0, \forall x$, if $0 < |x - 2| < \delta/4$ then $|3x - 6| < \epsilon/7$.

Remark: This comes from homework problem 2, p49, about an alternative definition of limit, applied to the example $\lim_{x\to 2} 3x = 6$. Your proof should be pretty similar to a standard limit proof, but you may need to replace the usual $\delta = \epsilon/3$ by something slightly new.

6) Choose ONE textbook proof.

a) Every convergent sequence is Cauchy.

b) The interval (0,1) is not countable (not denumerable).

c) If a sequence is monotone and bounded then it converges.

7) [20pts] Choose TWO to prove, circle your choices, and label your two answers clearly. If you answer on the back, leave a note here as usual. Use the usual definition(s) of limit (use ϵ, etc).

a) $\lim_{x\to 3} 2x^2 = 18$.

b) $\lim_{x\to\infty} \frac{4}{x+4} = 0$.

c) $\lim_{x\to 1} \frac{|x|}{x} = 1$.

Bonus [5 pts]: Find a 1-1 correspondence $f : [0, 1] \to (0, 1)$. Or, for a little partial credit, prove that f exists without actually finding an example.

Remarks and Answers: The average among the top 18 was approx 73 out of 100, which is fairly good. The high scores were 96 and 91. The results were generally good on the proof problems 5, 6 and 7, but rather low on problems 1 and 4. Here is an advisory scale for Exam 2:

A's 80 to 100
B's 70 to 79
C's 60 to 69
D's 50 to 59

I think the drop date is Oct. 29. To estimate your semester grade average your two exam scores and use that with the scale below. It is hard to predict the exact effect of the HW at this time, so that was not included in forming the scale. You can email me (before approx 2pm Monday) for more information or advice.

A's 79 to 100
B's 69 to 78
C's 59 to 68
D's 49 to 58

1a) $A \preceq B$ means $\exists f : A \to B$ which is 1-1. $A \prec B$ means $A \preceq B$ and $A \not\sim B$.

1b) Define an eq.rel on $A = Z \times Z \setminus \{0\}$, that $(a, b) \sim (c, d)$ means $ad = bc$. Then $Q = A/\sim$

(the eq.classes).

1c) Two systems S, T are considered the same if there is an *isomorphism* $f : S \to T$ between them. The f must be a 1-1 correspondence that preserves operations, order, etc. We say that our R (the field we constructed) is the unique C.O.F. to mean that any other C.O.F is isomorphic to our R. I gave full credit if you included the word *isomorphism*.

Another example: Technically $N \subset Z$ is not true, based only on the constructions we did, because we constructed these two systems in different ways. But it is true that Z has a subset S which is isomorphic to N. For example, the number we usually refer to as '3' corresponds to the element $\sigma(\sigma(\sigma(0)))$ defined in the construction of N. It corresponds to the equivalence class $[(3,0)]_\sim$ defined while constructing Z. Hopefully you see the isomorphism from these comments. For practical purposes, $N \subset Z$ is true and it is harmless to write formulas like $\sigma(\sigma(\sigma(0)))) = 3 = [(3,0)]_\sim$. For more examples, which are not number systems, review isomorphism of graphs and trees from Discrete Math.

d) It is not possible to get two different answers for a sum or product, etc, by working from two different representatives of an equivalence class. For example, $[3,1]_\sim + [4,5]_\sim$ should be equal to $[13,11]_\sim + [4,5]_\sim$ in Z.

2a) Every nonempty subset of R that is bounded above has a lub.

2b) It was used to define multiplication on N. 20.

3) FFTTT FFTFF

4) The best simplest answer seems to be 'One interval is empty or contains the lub or glb of the other.' There may be other correct answers. I gave partial credit for various sufficient conditions, such as:

a) If $I \cap J \neq \emptyset$ (that is, they are not disjoint) then $I \cup J$ is an interval.

b) If either set is empty, then $I \cup J$ is an interval.

We should also include examples like $[0,1) \cup [1,2] = [0,2]$, but most people didn't consider this or didn't describe it precisely.

5) Fix $\epsilon > 0$. Set $\delta = 4\epsilon/21$. Fix x and assume $0 < |x-2| < \delta/4$. So $|3x-6| = 3|x-2| < 3\delta/4 = \epsilon/7$.

Most answers that included $\delta = 4\epsilon/21$ were good proofs. Some answers (that did not start with something about $\epsilon > 0$, or used a different δ) were wrong or impossible to follow. You did not have to mention $f(x)$, a or L since the question was not about a limit, at least not on the surface.

6) Most people chose 6b and did OK, perhaps by memorizing Cantor's proof. The most common problem was a failure to explain the contradiction at the end. It is based on these 3 steps, which should have appeared earlier in your proof, with explanations:

$(0,1) = \{x_1, x_2, \ldots\}$
$x \in (0,1)$
$x \notin \{x_1, x_2, \ldots\}$

The first two are easy, but were often neglected, or had serious notational problems. The third is perhaps the main point and most people handled that fairly well.

7a) Everyone chose to do 7a, though it may be the hardest. Most people did well, especially if the scratch work led to δ =min $\{\epsilon/14, 1\}$. The most common problems were with handling the $x+3$ factor. Some good scratch work for $x+3$ might look like this:

Note to self for later: make sure $\delta \leq 1$. Then, if $|x-3| < \delta \leq 1$ we get $-1 < x-3 < 1$. Adding 6 (to get $x+3$ into this), we get $5 < x+3 < 7$ so that $|x+3| < 7$. [End of this phase of the scratch work. Most people who did this correctly also got to δ =min $\{\epsilon/14, 1\}$, and a pretty good proof.]

7b) was similar to 7a) in popularity and success. You can use $N = 4/\epsilon$ or $N = 4/\epsilon - 4$. A common minor mistake was to assume $x + 4 > 0$ (to get $x + 4 = |x + 4|$ for example) without any comment.

7c) Only one person chose this, maybe because it is unusual, but it is the simplest. The main idea: set $\delta = 1$ to get $x > 0$, so that $|f(x) - L| = |1 - 1| = 0 < \epsilon$.

B) Let a_n be some sequence starting with 0, such as 0, 1/2, 1/3, 1/4 .... Let b_n be some sequence disjoint from the range of a_n, starting with 1, such as 1, 2/3, 3/4, 4/5 .... Define f on the range of a_n by $f(a_n) = a_{n+1}$. So, for example, $f(0) = 1/2$. Define f on the range of b_n by $f(b_n) = b_{n+1}$. So, for example, $f(1) = 2/3$. For all other $x \in [0,1]$, let $f(x) = x$.

Then f is defined on $[0,1]$ and is 1-1 (because it is the union of three 1-1 functions with disjoint ranges). It is onto $(0,1)$ (because this is the union of the 3 ranges). So, $f : [0,1] \to (0,1)$ is a bijection. Done.

The easier proof for partial credit is based on Schroder-Bernstein. We just need two injections, going both ways, such as $g : [0,1] \to (0,1)$ defined by $g(x) = x/2 + 0.001$ and $h : (0,1) \to [0,1]$ defined by $h(x) = x$.