MAA 3200 | Nov 17, 2003
Exam III and Key | Prof. S. Hudson

---

1) Short answer (5 pt each)

a) Let $E = \{x \in R : 5x \in N \text{ and } x > 2\}$. Find inf E.

b) Let $U = Q$. Give a counterexample to the completeness property. [Define the set E, but you do not have to prove anything].

c) Give a counterexample to the Nested Interval Theorem when the intervals are not required to be closed.

d) State the Archimedean Principle.

2) [20pts] Answer TRUE or FALSE; you do not have to justify your answers.

An increasing sequence must either converge in R, or diverge to $+\infty$.

Every bounded sequence has a Cauchy subsequence.

Every bounded set has a supremum.

If $\forall n \in N, 0 < x_{n+1} < x_n/2$, then $x_n \to 0$.

$\forall M, \exists N$, such that if $n \geq N$ then $n > M$.

$|x_n| \to 0$ if and only if $x_n \to 0$.

If $|x_n|$ converges then x_n converges.

If sup $E = b > 0$ then $E \cap (0, b) \neq \emptyset$.

If $\forall n \in N, x_{n+1} \leq x_n \leq y_n \leq y_{n+1}$ then $\exists x \in R, \forall n \in N, x_n \leq x \leq y_n$.

If A and B are uncountable subsets of R then $A \cap B$ is also uncountable.

3) [15 pts] Assume that F is an ordered field (so, it satisfies Postulates 1 and 2 in Wade, so it contains 0 and 1, which aren't equal, etc). Prove that $0 < 1$.

4) [15 pts] Prove that $3(1 + 1/n) \to 3$ using the definition of limit (using ϵ).

5) [15 pts] Let $x \in R$. Show that there is a sequence of rationals $r_n \to x$.

6) [15 pts] Choose ONE proof:

a) Every Cauchy sequence is bounded.

b) If a Cauchy sequence has a convergent subsequence, it converges.

c) State and prove the Monotone Convergence Theorem (the increasing part only).

Brief Answers: Average = about 62/100. The proofs were mostly OK, but the results weren't so good on problems 1-2.

1a) $x = 11/5$. [This set is like the one in the proof of the density theorem. It includes the numbers 11/5, 12/5, 13/5 etc. Just note the pattern and choose the first term.]

1b) Let $E = \{x \in Q | x^2 < 2\}$. If $U = R$, then lub $E = \sqrt{2}$, but the lub doesn't exist in Q.

1c) Let $I_n = (0, 1/n)$ so that $\cap I_n = \emptyset$. See Remark 2.24 on page 46.

Suggestion: When you read a math book, focus on the theorems, and test each hypothesis as the author does in remarks 2.24 and 2.25. This helps you understand what each theorem is about, so you know when to use it. It also helps with exam questions like 1b and 1c (examples may be hard to find during a test).

1d) See text.

2) **TTFTT TFFTF** See me, if needed. For Part 2, combine Theorems 2.26 and 2.29. Part 8; any set E with just one point forms a counterexample. Part 9 is just the nested interval theorem.

3) We are given trichotomy and that $0 \neq 1$, so either $0 < 1$ or $0 > 1$. If $0 < 1$ we are done. If $0 > 1$ then by the multiplicative property, $1 \times 0 < 1 \times 1$. So, $0 < 1$, and we are done.

4) Let $\epsilon > 0$. Let $N > 3/\epsilon$. Assume $n \geq N$. Then $|(3(1 + 1/n) - 3| = |3/n| = 3/n \leq 3/N < \epsilon$. Done.

5) Use the density theorem to get $r_n \in (x - 1/n, x + 1/n)$ and then apply the Squeeze theorem. See HW key.

6) See text. Parts a) and b) are both contained in the proof of Thm 2.29 [so, you should not quote the fact that Cauchy sequences converge as part of your proof].