MAA 3200 Nov 16, 2010
Exam III and Key Prof. S. Hudson

1) [20 pts] Answer True or False. You do not have to justify these. Let $A = \cap_{n=1}^{\infty} I_n$, where I_n are intervals in the real line; use this in the first five.

If the I_n are open then A is open.
If the I_n are bounded then A is bounded.
If the I_n are open and nested ($I_{n+1} \subseteq I_n$), then A contains a rational number.
If the I_n are closed then A is nonempty.
If A contains 3 and 5, then A contains 4.
Every non-empty subset of the extended real number system has a least upper bound.
The set of all polynomials (of arbitrary degree) with integer coefficients is countable.
Every infinite subset of the interval $[0, 2\pi]$ has a limit point.
The set of positive rational numbers is dense in the real numbers.
Any set $S \subseteq R$ with at least one interior point has at least 10 interior points.

2) [15 pts] Let $A = \{(x, 0) \in R^2 : x \in R^1\}$ (the x-axis). Show A is closed in R^2 (using the usual metric).

3) [15 pts] Explain briefly why Z_4 is not a field.

4) [15 pts] Define the set of integers Z from N as done in class, using equivalence classes.

5) [15 pts] Define $\rho : R^3 \times R^3 \to R^3$ by $\rho(\mathbf{x}, \mathbf{y}) = \max \{|x_1 - y_1|, |x_2 - y_2|\} + |x_3 - y_3|$ (where $\mathbf{x} = (x_1, x_2, x_3)$). Show that this satisfies the triangle inequality (part 3 of the definition of metric). You can use any familiar formulas about $|a - b|$, etc.

6) [15 pts] Choose ONE proof:

a) Every nonempty open interval (a, b) in R^1 contains a rational number.

b) If G_j are open sets in M, then $A = \cup_{j=1}^{\infty} G_j$ is open.

c) The complement of an open set G in M is closed.

7) [5 pts] With the notation we used to develop the natural number system, find $\sigma(s_3(7))$.

Bonus 1 [approx 5 pts]: Suppose that I_n are half-open intervals in R^1. If n is odd, then $I_n = [a_n, b_n)$. If n is even, then $I_n = (a_n, b_n]$. For all n, $I_{n+1} \subseteq I_n$. What can you say about $A = \cap_{n=1}^{\infty} I_n$? Must it be open, or closed ? Must it be nonempty ?

Bonus 2 [approx 3 pts]: Briefly, what do you think of the books by a) Velleman, b) Morash and c) Shilov ? There is no 'correct' answer, but convince me you have studied from at least 2 of the 3, and can see some differences.

Remarks and Answers: The average was about 65 / 100, which is fairly normal. The results were OK on all problems except problem 2). The highest grade was 97. A rough scale is:

A's 75 - 100
B's 65 - 74
C's 55 - 64
D's 45 - 54

To estimate your semester grade, so far, I added your 3 exam scores, but not your HW yet. I scaled that [A's starting at around 195 / 300, with the other grades 30 points apart] and wrote the result on the upper right of your exam. If your HW average is very different from your exam average, you should adjust this. For example, if you have 165 / 300 on exams, I called that a B-, but if you have not handed in any HW, your grade is more like a C-.

To estimate what the final might do for you, a very optimistic hope is an exam grade about 30 points above your current average, and that would raise your current average grade by one letter. Example 1: with 165 / 300 your average is 55, and you might hope for an 85, which should bring you from a B- to an A-. Example 2: if you have an F now, you have almost no chance to bring that up to a C. For most people, the final won't change their average very much, and the chances of going up or down depend mainly on their study efforts.

1) FTFFT TTTFT (see me about any unclear TF)

2) The direct route is to show that every limit point of A is in A [using def of closed], and this may be easiest from the contrapositive. So, assume $x \notin A$ (ETS x is not a limit point). Since $x = (x_1, x_2) \notin A$, we know $x_2 \neq 0$. Wlog $x_2 > 0$. Set $\epsilon = x_2$; ETS $B_\epsilon(x)$ contains no points of A. Suppose it contains some $a \in A$ (to get a contradiction). Note that $a_2 = 0$, so $x_2 = |x_2| = |x_2 - a_2| \leq ((x_1 - a_1)^2 + (x_2 - a_2)^2)^{1/2} = \rho(x, a)$. Since $a \in B_\epsilon(x)$, we get $\rho(x, a) < \epsilon = x_2$, a contradiction to the previous sentence.

Another strategy is to show that A^c is open, but the details are almost the same as above; eg show that any point x not in A is an interior point of A^c (which is equivalent to what we did above).

3) In a field, every $x \neq 0$ has a mult inverse. But $2 \in Z_4$ does not. Justify this briefly by checking all elements, or explaining, perhaps based on $2|4$. I gave partial credit for saying that 4 is not prime, but this reasoning is based on an idea which hasn't been proved in our class.

4) Briefly, $A = N \times N$, $(a,b) \sim (c,d)$ means $a + d = b + c$, and $Z = A/\sim$.

5) ETS $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Since there are 3 'max's here, it is tempting to split the proof into $2^3 = 8$ cases, or to take some shortcuts using 'wlog's. Some students used 3 wlog's, which is too much; I think 1 wlog is best:

Proof: Wlog $|x_1 - z_1| \geq |x_2 - z_2|$, so that $\rho(x,z) = |x_1 - z_1| + |x_3 - z_3|$. From the R^1 triangle inequality, $|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$ and $|x_3 - z_3| \leq |x_3 - y_3| + |y_3 - z_3|$. Also by the def of max, $|x_1 - y_1| \leq \max\ \{|x_1 - y_1|, |x_2 - y_2|\}$ and $|y_1 - z_1| \leq \max\ \{|y_1 - z_1|, |y_2 - z_2|\}$. So (by simple algebra) $\rho(x,z) \leq \max\ \{|x_1 - y_1|, |x_2 - y_2|\} + |x_3 - y_3| + \max\ \{|y_1 - z_1|, |y_2 - z_2|\} + |y_3 - z_3| = \rho(x,y) + \rho(y,z)$.

6) These were textbook proofs, and most people did well on them.

B1) The intersection is closed and non-empty (stating this was worth a few points of partial credit). Nobody gave a careful proof of either of these, and I might still give a little extra credit to the first student who can do so.