MAA 3200 Nov 26, 2013
Exam III and Key Prof. S. Hudson

1) [20 pts] Answer True or False. Assume that S and T are non-empty, bounded sets in R, and that $(G, *)$ is a group. You do not have to justify.

If G is isomorphic to an Abelian group H, then G is also Abelian.

If G is Abelian, then $*$ is commutative and associative.

glb $S <$ lub S.

The set $\{-1, 1\} \subset R$, with multiplication, forms a group.

If $S \subseteq [3, 5]$ then S has an accumulation point x, with $x \leq 5$.

If $L =$lub rng (x_n) exists, then $\lim x_n = L$.

In Q, every Cauchy seq converges.

In Q, every Cauchy seq is bounded.

lub $(S \cap T) =$ min { lub S, lub T}.

If x is an accumulation point of S then some sequence $x_n \in S$ converges to x.

2) [10 pts] Choose ONE. Note that 2B is a bit harder and may net 5 extra points.

2A) [10 pts] Show that if $\forall n, x_n \geq 0$ and $\exists L = \lim x_n$ then $L \geq 0$.

2B) [15 pts] Show that the product of two Cauchy sequences is also Cauchy. You can use the theorem that Cauchy implies bounded (in R).

3) These are short answer problems, 5 pts each. Show enough work or reasoning, but you do not have to prove your answers, unless asked.

a) Let $x_n = 3 + \frac{(-1)^n n}{n+2}$. Compute $\limsup x_n - \liminf x_n$.

b) Define x_n as in (a). Let z_n be the subsequence x_{2n}. Let $y_n = x_n + 1$. Find all the accumulation points of $S =$ rng $(x_n) \cup$ rng $(z_n + y_n)$.

c) Give an example of a Cauchy sequence of rational numbers that converges to π in R.

d) Let $(G, *)$ be a group with identity elements e and e'. Prove that $e = e'$.

e) Let $A = \{q \in Q : q^2 < 1\}$ and $B = \{n \in Z : n^2 < 10\}$. Let $C = \{a + b : a \in A, b \in B\}$. Find lub C (in R, of course).

4) [10 pts] The table below defines an operation $*$ on the set $X = \{a, b, c, d\}$. Please label the columns with the letters a,b,c,d in that order (this is hard to typeset) and do the same for the rows. Show that $(X, *)$ is a group, and is isomorphic to $(Z/4, +)$. Your answer should include a definition of *isomorphism*, and an example of one, and 'some checking'

(just enough to show that you know what has to be checked).

$$\begin{pmatrix} c & a & d & b \\ a & b & c & d \\ d & c & b & a \\ b & d & a & c \end{pmatrix}$$

For a little extra credit - how many different isomorphisms are possible here ?

5) [10 pts] Choose ONE textbook proof:

a) State and prove the BW Thm, about acc. pts.

b) State and prove the completeness theorem, about Cauchy sequences in R.

6) [7 pts] Define Q, and $\cdot$, as done in class. Use that to show every nonzero element of Q has a multiplicative inverse (include existence of an identity element).

7) [8 pts] Define R and $+$, as done in class and prove $+$ is *well-defined*.

8) [10 pts] Choose ONE:

a) Define $x < y$ in R and prove $<$ is transitive (using Cauchy sequences, of course).

b) State the trichotomy principle and prove it in Z (using any needed results in N).

c) Outline the proof of the Lub Thm in R (as in my online pdf) and do one of the steps (these were assigned as HW).

Bonus [approx 5 pts]: Assume x_n and y_n are bounded. Prove or disprove: $\liminf x_n + y_n = \liminf x_n + \liminf y_n$.

Remarks and Answers: The average was 62 / 100 based on the top 7. There were 3 scores in the 70's, but only one in the 50 to 69 range. The average on each problem was OK, except for problems 5 and 7 and the Bonus. The rough scale is:

A's 70-100
B's 60-69
C's 50-59
D's 40-49

Your semester average is in the upper right, based only on the 3 exams. Please check that. The average for that stat is 64, with a high of 72. Scale:

A's 71-100
B's 61-70
C's 51-60
D's 41-50

1) TTFTF FFTFT

2) See me or the text. The key steps near the end of the expected proof of 2B are

$$|x_n y_n - x_m y_m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \le |x_n - x_m|\ |y_n| + |x_m|\ |y_n - y_m| < \epsilon/2 + \epsilon/2$$

One person took an unexpected shortcut, avoiding ϵ; Cauchy implies convergence, and the product of convergent sequences converges, etc. I reluctantly gave 10 points, but this was not fully acceptable because it uses a theorem more advanced than the one being proved (because this problem is used to construct the real numbers, for example). This is a fairly common problem with proofs in advanced classes. If in doubt, ask your prof which theorems are allowed as tools in the proofs.

3a) 2

b) 2,4,7,9

c) 3, 3.1, 3.14, etc

d) $e = e * e' = e'$ with explanation. I did not accept "$x * e = x * e'$ implies $e = e'$" This reasoning is not always valid, if $x = 0$ for example. It might work, *with additional explanation* , but nobody did that. Likewise, the formula $e' * e = e * e'$ might work, *with additional explanation*, but it is not quite acceptable standing alone. We are not given that $*$ commutes.

e) 4

4) The main point here (which many people missed) is to write out a bijection $f : G \to Z/4$, which is also an isomorphism. Partial credit for showing that you understand the vocabulary.

Set $f(b) = 0$ since b is clearly the identity element. Set $f(c) = 2$ since $c * c = b$ and $2 + 2 = 0$. Set $f(a) = 1$ and $f(d) = 3$ (or $f(a) = 3$ and $f(d) = 1$. This answers the bonus; exactly 2 isomorphisms are possible).

This is clearly 1-1 and onto. Check that f is also a homomorphism (it respects the operations) by checking $c * c = b$ is consistent with $2 + 2 = 0$, etc. Just check a few - it would be tedious to check everything. This indirectly implies G is a group (associative with inverses); just a few remarks on such reasoning is enough for this question.

5) See the text or lectures. These were advertised before the exam, so you should have been ready for at least one of these! In general, any named theorem (such as BW) is worth careful study. For most people, that means memorizing the statement, reviewing the proof until it makes sense, and memorizing any non-routine steps of the proof. In the BW proof, for example, the key ideas are

splitting intervals, over and over
using their endpoints, and the lub theorem, to define x
using ϵ methods to show x is an acc. pt.

6) See the pdf posted online for the definitions. You weren't expected to memorize them word-for-word, but you should remember the equivalence relation on $Z \times Z^*$, etc.

The id elt is $[(1,1)]$ (no proof required, though it is easy). Also, $[(a,b)] = 0$ iff $a = 0$ (no proof required, though it is easy). Given a nonzero elt $[(a,b)]$ we must produce an inverse. Since $a \neq 0$ we know $[(b,a)] \in Q$, and $[(a,b)] \cdot [(b,a)] = [(ab, ba)] = [(1,1)]$.

Most people wrote something like $(a,b) \cdot (b,a) = 1$ (not the best notation, and not really a proof by itself, but it is the main idea).

7) See the pdf posted online for the definitions. 'Well-defined' means: if $[x_n] = [r_n]$ and $[y_n] = [s_n]$ then $[x_n] + [y_n] = [r_n] + [s_n]$. By the definition of $+$, ETS: $[x_n + y_n] = [r_n + s_n]$, which means $\lim\ (x_n + y_n - (r_n + s_n)) = 0$.

We are given that $\lim\ (x_n - r_n) = 0$ and $\lim\ (y_n - s_n) = 0$. The limit-of-a-sum theorem leads to $\lim\ (x_n + y_n - (r_n + s_n)) = 0$. Done.

8) Most of these were HW problems from the pdfs. Part a) is the shortest, but involves an ϵ or two. See me or our LA if you need help.

B) False. Counterexample: let $x_n = (-1)^n$ and $y_n = -x_n$. Then $\liminf x_n + y_n = \liminf 0 = 0$ but $\liminf x_n + \liminf y_n = -1 - 1 = -2$.