MAA 3200 Dec 7, 2010

Final Exam and Key Prof. S. Hudson

1) Let $f(x) = x^2$ on R. Let $E = [-1, 4]$. a) Find $f(E)$. b) Find $f^{-1}(E)$.

2) Show by induction that if $n > 0$ is odd, then $2^n + 3^n$ is a multiple of 5.

3) Suppose $x \in A \subset R$ is also an upper bound of A. Show that $x = \text{lub } A$ (which is the same as $\sup A$).

4a) State the Bolzano-Weierstrass theorem, about bounded sets in R. 4b) Show by example that it fails in the metric space Q of rational numbers. [If you just can't remember this theorem, you can replace it by the Nested Interval Theorem, in both a) and b), for partial credit].

5) Give examples of closed sets $S_n \subset R^2$ such that $\cup_{n=1}^{\infty} S_n = B_1(0)$ (the open unit ball).

6) Give ONE of these textbook proofs:

a) Every Cauchy sequence in R converges.

b) R is not countable.

7) Prove that $\lim_{(x,y)\to(2,3)} 2xy = 12$ using the usual metric on R^2. [For partial credit, you can state the definition of limit needed here, and the formula for the usual metric on R^2].

8) Suppose $\{x_n\}$ is increasing and bounded in R. Show that the sequence converges (to its lub).

9) [20 pts] Answer True or False:

[temporarily missing - let me know if you need these]

Bonus) Suppose $E \subset R^2$ is closed and $x \notin E$. Prove that $\inf_{a \in E} \rho(x, a) > 0$.

Remarks and Answers: The average was about 60/100, with a high of 85. The unofficial scale should be similar to that of Exam 3. The results were OK on most problems, except 8), and with a slight dip on page 2. Not many people got 7) completely right, but most got partial credit by following the directions.

Good luck in your future courses ! I'll probably be in touch with some of your instructors, such as in MAA 4211, and hope to hear how that goes. You are welcome to visit me in the Spring term.

1) $f(E) = [0, 16]$, $f^{-1}(E) = [-2, 2]$. Of course, this refers to the pre-image of the set, not to the inverse of the function (see Velleman approx Ch 5).

2) Basis Step: Let $n = 1$ and note that $2^1 + 3^1 = 5$ is a multiple of 5.

Ind Step: Assume n is odd and that $5|\ 2^n + 3^n$. Since $n+2$ is the next odd integer, ETS $5|\ 2^{n+2} + 3^{n+2}$. [If this modification of induction bothers you, then you can set $n = 2k+1$, and use induction on k, but it is the same thing].

$$2^{n+2} + 3^{n+2} = 4 \cdot 2^n + 9 \cdot 3^n = 4 \cdot [2^n + 3^n] + 5 \cdot 3^n$$

and both summands are multiples of 5. Done.

3) We must show x is the *least* ub. Suppose $y < x$. Since $x \in A$ this implies y is not an upper bound of A. So, no upper bound of A can be smaller than x. Done.

4b) Let $S = \{1, 1.4, 1.41, \ldots\} \subset Q$, which we discussed often. It has no limit point in Q.

5) Let $S_n = \overline{B_{1-1/n}(0)}$, a closed ball slightly smaller than the unit ball. We did several examples just like this, but more in R^1 than R^2. Several people had the idea [mostly] but gave examples of intervals in R^1.

6) See text.

7) Scratch work: Let $A = |2xy - 12| = |2xy - 4y + 4y - 12| \le |2y(x-2)| + 4|y-3|$. We'll want this $< \epsilon$ and will want some bound such as $|y| < 4$ (since $y \approx 3$ this seems plausible, if $\delta < 1$, or so).

Proof: Let $\epsilon > 0$. Set $\delta = \min\ \{\epsilon/16, 1\}$. Assume $\rho((x,y),(2,3)) < \delta$. Note that $|x-2| \le \rho((x,y),(2,3)) < \epsilon/16$, and similarly, $|y-3| \le \rho((x,y),(2,3)) < \epsilon/16$. Also, $|y-3| < 1$ which implies, by algebra, that $|y| < 4$. Combining all this, we get $A < \epsilon$, done. [I've left several small gaps for you to fill in].

8) Let $L =$lub $\{x_n\}$, so ETS $x_n \to L$. Let $\epsilon > 0$. Since $L - \epsilon < L$ it is not an upper bound, and $\exists x_N > L - \epsilon$. If $n > N$ then $x_n \ge x_N$ (by increasing) so $L - \epsilon < x_n \le L$ (by transitivity and that L is an upper bound). This implies $|x_n - L| < \epsilon$, done.

I don't think there are many other ways to do this one. It is false in Q, so you have to use some kind of completeness theorem, such as the lub axiom [some people tried using Cauchy-implies-Converges instead, but didn't quite make this idea work). If you use the lub idea, don't forget that L is the *least* ub; you can't write a proof without that.

B) Since E^c is open, x is an interior point of that, and $\exists \epsilon > 0$ such that $B_\epsilon(x) \subset E^c$. The contrapositive tells us that if $a \in E$ then $\rho(x,a) > \epsilon$. So, ϵ is a lower bound for these numbers, and the inf [glb] is $\ge \epsilon > 0$.